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A Straightening Algorithm for Row-Convex Tableaux

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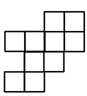
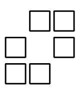
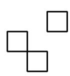
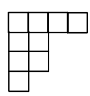
We produce a new basis for the Schur and Weyl modules associated to a row-convex shape D. The basis is indexed by a new class of "straight" tableaux which we introduce by weakening the usual requirements for standard tableaux. Spanning is proved via a new straightening algorithm for expanding elements of the representation into this basis. For skew shapes, this algorithm specializes to the classical straightening law. The new straight basis is used to produce bases for flagged Schur and Weyl modules, to provide Groebner and SAGBI bases for the homogeneous coordinate rings of some configuration varieties, and to produce a flagged branching rule for row-convex representations. Systematic use of supersymmetric letterplace techniques enables the representation theoretic results to be applied to representations of the general linear Lie superalgebra as well as to the general linear group.

1. INTRODUCTION

Akin *et al.* [ABW82] give a construction that associates a GL_n-representation to any generalized shape like

 or  or  or  (1)

Significant progress has been made by Reiner and Shimozono and by Lakshmibai and Magyar in describing bases for these GL_n-representations and for the associated flagged representations of the Borel subgroup of lower triangular matrices in GL_n. As one expects, these bases are indexed by some subset of the generalized tableaux found by filling each cell in the generalized shape with a number from 1 to n.

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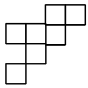
The present paper shows how to construct a well-behaved *straight* basis for the representations associated to any *row-convex* shape, such as

,

with no gaps in any row. In particular, we give a local condition for testing whether a tableau is straight, we give a straightening law that modifies only two rows at a time, and the basis we present reduces immediately to a flagged basis. The straight basis is distinct from the bases produced by Reiner and Shimozono and Lakshmibai and Magyar, but see [T00a] for some combinatorial and algebraic relationships between these bases. The straight bases provide a canonical choice of basis for certain row-convex and column-convex "almost-skew shapes." These shapes were shown by Woodcock [W94] to possess a class of easily flagged bases, but no method was presented for distinguishing a basis in this class or for straightening elements of the representation into a linear combination of basis elements.

Results on flagged tableaux are deduced in Section 6 from the main theorem on straight bases. As shown in Section 7, the straight basis and straightening algorithm may be applied to produce quadratic Groebner bases and SAGBI bases for the homogeneous coordinate rings of certain configuration varieties. Further applications to commutative algebra may be found in [T97a, T00b]. Applications to the representation theory of GL_n, B_n, S_n, and the general linear Lie superalgebras are derived in Section 8 where a branching rule is produced for decomposing a row-convex GL_n-representation in terms of GL_{n-1}-representations.

This paper studies the Schur and Weyl modules as special cases of the super Schur modules which we construct as submodules of the letterplace superalgebra. All results in this paper are characteristic-free and the requisite background on superalgebras is detailed in Section 2. Much of the presentation in Section 2 is new and, we hope, accessible to the non-specialist. The construction proper is given in Section 3. Straight tableaux are introduced and independence is proved in Section 4. Section 5, the heart of the paper, presents the straightening algorithm.

2. POLYNOMIAL SUPERALGEBRAS

This section introduces the definitions required to make the main results of this paper characteristic free and applicable to Weyl modules. The reader concerned only with Schur modules in characteristic 0 may safely take $\mathscr{L}$ and $\mathscr{P}$ to be the positive integers, $\mathbf{N}$ (or finite subsets of $\mathbf{N}$).

The set $\mathscr{L}$ may then be thought of as indexing the rows of a generic matrix $(x_{i,j})$ and $\mathscr{P}$ indexes the columns. We may then take *Super*$([\mathscr{L} | \mathscr{P}])$ to be the polynomial ring whose variables are matrix entries $x_{i,j}$. The *letterplace* $(i \mid j)$ is taken to be shorthand for $x_{i,j}$ and the expression $(-1)^{\binom{k}{2}}(i_1, \dots, i_k \mid j_1, \dots, j_k)$ is taken to be the determinant of the $k \times k$ minor (x_{i_r, j_s}) of the matrix $(x_{i,j})$.

The constructions used in this paper take place inside polynomial superalgebras over the integers, $\mathbf{Z}$, that is, inside tensor products of symmetric, exterior, and divided powers algebras. We construct the polynomial superalgebras over $\mathbf{Z}$ as $\mathbf{Z}$-subalgebras of a symmetric algebra over the rationals, $\mathbf{Q}$, tensored with an exterior algebra over $\mathbf{Q}$. Write the free symmetric and exterior $\mathbf{Z}$-algebras generated by a set $\mathscr{L}$ as *Sym*$(\mathscr{L})$ and $\Lambda(\mathscr{L})$. These are $\mathbf{Z}$-subalgebras of the symmetric and exterior $\mathbf{Q}$-algebras $Sym_{\mathbf{Q}}(\mathscr{L})$ and $\Lambda_{\mathbf{Q}}(\mathscr{L})$ associated to $\mathscr{L}$. The divided powers algebra, *Div*$(\mathscr{L})$ of a set $\mathscr{L}$ is the $\mathbf{Z}$-subalgebra of $Sym_{\mathbf{Q}}(\mathscr{L})$ generated by all $x^i/i!$ for all $x \in \mathscr{L}$.

We define a *signed set* to be a set $\mathscr{L}$ together with a function $| \ | : \mathscr{L} \to \mathbf{Z}_2$. We say that elements in the preimage of 0 are *positively signed*; we call this preimage $\mathscr{L}^+$. Elements in the preimage, $\mathscr{L}^-$, of 1 are said to be *negatively signed*. A signed set $\mathscr{L}$ endowed with a total order, $<$, is called a (signed) *alphabet*. For notational convenience, we define two transitive binary relations, $<\!\!+$ and $<\!\!-$ on $\mathscr{L}$. We say that $a <\!\!+ b$ (respectively $a <\!\!- b$) when $a < b$ or when $a = b$ and $|a| = |b| = 0$ (respectively $a = b$ and $|a| = |b| = 1$.) We define the relation $a +\!\!> b$ (respectively $a -\!\!> b$) when $b <\!\!+ a$ (respectively $b <\!\!- a$.) Alternately, we may characterize $a +\!\!> b$ and $a -\!\!> b$ as $a \not<\!\!- b$ and $a \not<\!\!+ b$, respectively.

A *superalgebra* is simply an algebra with a $\mathbf{Z}_2$-grading. We construct a $\mathbf{Q}$-superalgebra with the elements of a signed set as generators and such that the grading on these generators is $| \ |$. For any signed set $\mathscr{L}$, define $Super_{\mathbf{Q}}(\mathscr{L})$ to be $Sym_{\mathbf{Q}}(\mathscr{L}^+) \otimes \Lambda_{\mathbf{Q}}(\mathscr{L}^-)$. Likewise, we define *Super*$(\mathscr{L})$ to be *Div*$(\mathscr{L}^+) \otimes \Lambda(\mathscr{L}^-)$; as above, we may consider this to be a $\mathbf{Z}$-subalgebra of $Super_{\mathbf{Q}}(\mathscr{L})$. Given another signed set $\mathscr{P}$, we will define the "letter-place" algebra, *Super*$([\mathscr{L} \mid \mathscr{P}])$, to be a $\mathbf{Z}$-subalgebra of $Super_{\mathbf{Q}}(\{x_{a,d}\}_{a \in \mathscr{L}, d \in \mathscr{P}})$ where $|x_{a,d}| = |a| + |d|$. In particular, *Super*$([\mathscr{L} \mid \mathscr{P}])$ is the subalgebra generated by all $x_{a,d}$ and by all $x^i_{a,d}/i!$ with $a, d \in \mathscr{L}^+ \times \mathscr{P}^+$, $i \in \mathbf{N}$. This algebra is naturally isomorphic to

$$\Lambda(\mathscr{L}^- \times \mathscr{P}^+ \uplus \mathscr{L}^+ \times \mathscr{P}^-) \otimes Sym(\mathscr{L}^- \times \mathscr{P}^-) \otimes Div(\mathscr{L}^+ \times \mathscr{P}^+).$$

We extend $| \ |$ to a $\mathbf{Z}_2$ grading of *Super*$([\mathscr{L} | \mathscr{P}])$. Following [GRS87], we write the elements $x_{a,d}$ of *Super*$([\mathscr{L} | \mathscr{P}])$ as the signed variables $(a \mid d)$, and we will define the *biproduct*, $(w_1, \dots, w_k \mid v_1, \dots, v_k)$, of a pair of sequences $\underline{w}$ and $\underline{v}$ in respectively $\mathscr{L}$ and $\mathscr{P}$.

DEFINITION 2.1. Given sequences $\underline{w} = w_1, \ldots, w_k \in \mathscr{L}$ and $\underline{v} = v_1, \ldots, v_k \in \mathscr{P}$, define $(\underline{w} \mid \underline{v}) = (w_1, \ldots, w_k \mid v_1, \ldots, v_k) = \sum_{\sigma \in S_k} (-1)^{n_\sigma} (w_{\sigma(1)} \mid v_1) \cdots (w_{\sigma(k)} \mid v_k)$ where

$$n_\sigma = \#\{(i,j) : i < j, \sigma^{-1}(i) > \sigma^{-1}(j), w_i, w_j \text{ are negative}\}$$
$$+ \#\{(i,j) : i > j \text{ and } w_{\sigma(i)}, v_j \text{ are negative}\}.$$

The following definition/proposition indicates that the biproduct can be thought of as a bilinear map on $Super(\mathscr{L}) \times Super(\mathscr{P})$.

DEFINITION 2.2. Given sequences $w = w_1, \ldots, w_k \in \mathscr{L}$ and $v = v_1, \ldots, v_k \in \mathscr{P}$, define $(w_1 \cdot w_2 \cdots w_k \mid v_1 \cdot v_2 \cdots v_k) = (w_1, w_2, \ldots, w_k \mid v_1, v_2, \ldots, v_k)$. Extend this by bilinearity to a map $(\,|\,) : Super(\mathscr{L}) \times Super(\mathscr{P}) \to Super([\mathscr{L} \mid \mathscr{P}])$.

It is straightforward to check that this map is well-defined.

In order to better handle divided powers as elements of a $\mathbf{Z}$-subalgebra contained in a $\mathbf{Q}$-algebra, we make the following definition. If $\underline{w}$ is a sequence in $\mathscr{A}$ for some signed set $\mathscr{A}$, then we define $\mathbf{c}(\underline{w})!$ to be $\prod_{i \in \mathscr{A}^+}$ (#times i appears in $\underline{w}$)!.

Call a monomial $M = \prod_i (l_i \mid p_i) \in Super_{\mathbf{Q}}([\mathscr{L} \mid \mathscr{P}])$ *sorted* when $p_1 \leq p_2 \leq \cdots$ and $p_i = p_{i+1} \in \mathscr{P}^+$ implies $l_i <\!+ l_{i+1}$ and dually $p_i = p_{i+1} \in \mathscr{P}^-$ implies $l_i -\!> l_{i+1}$. $Super([\mathscr{L} \mid \mathscr{P}])$ is a free $\mathbf{Z}$-module with basis consisting of the *divided powers monomials* $\{\frac{1}{\mathbf{c}(M)!} M\}$ for all sorted monomials $M \in Super_{\mathbf{Q}}([\mathscr{L} \mid \mathscr{P}])$. Here $\mathbf{c}(M)! = \mathbf{c}((l_1 \mid p_1), (l_2 \mid p_2), \ldots)!$. We will consider two monomials (respectively divided powers monomials) the same when they differ by a nonzero scalar multiple (respectively a multiple of ± 1.)

Define a function $\mathrm{Tab}(\underline{w} \mid \underline{v})$ when $\underline{w}$ (respectively $\underline{v}$) is a k-tuple of letters in $\mathscr{L}$ (respectively $\mathscr{P}$) by

$$\mathrm{Tab}(w_1, \ldots, w_k \mid v_1, \ldots, v_k) = \frac{1}{\mathbf{c}(\underline{w})!\, \mathbf{c}(\underline{v})!} (-1)^{\#\{(i,j) : i > j, w_i \in \mathscr{L}^-, v_j \in \mathscr{P}^+\}} (\underline{w} \mid \underline{v}).$$

Observe that the divided powers monomials occur with coefficient ± 1 in the expansion of $\mathrm{Tab}(\underline{w} \mid \underline{v})$ and that if $w_1 <\!+ w_2 <\!+ \cdots <\!+ w_k$ and $v_1 <\!+ v_2 <\!+ \cdots <\!+ v_k$, then the basis element $(1/\mathbf{c}(\prod_i (w_i \mid v_i))!) \prod_i (w_i \mid v_i)$ appears with coefficient 1.

3. SCHUR MODULES, WEYL MODULES, AND GENERALIZATIONS

In this section, we define our primary object of study, the super-Schur module as a **Z**-submodule of a letterplace algebra. The unsigned cases produce the Schur and Weyl modules of Akin *et al*. [ABW82] when $\mathscr{L} = \mathscr{L}^-$ and $\mathscr{L} = \mathscr{L}^+$, respectively.

We define two tableaux associated to a shape. The first is useful for referring to cells in the shape and the second plays a fundamental role in our construction of the super-Schur modules.

DEFINITION 3.1. Let D be a shape.

Define $F(D)$ to be the tableau of shape D whose cells are labeled $1, 2, 3, \ldots$ starting with the northmost cell in the leftmost column and continuing down the column, then down the second leftmost column, etc. In this paper, the signs of the letters in $F(D)$ are irrelevant.

A tableau of shape D is termed *Deruyts* if it is obtained by filling each cell in the diagram with the cell's column index viewed as a negative variable. We denote such a tableau by $Der^-(D)$.

Shapes appearing in this paper are assumed, unless otherwise noted, to have first coordinate 1 *in their top rows and second coordinate* 1 *in their leftmost columns.*

EXAMPLE 3.1.

$$F\left(\begin{array}{cccc} & & \square & \square \\ \square & & \square & \square \\ & & \square & \\ \square & & & \end{array}\right) = \begin{array}{cccc} & & 3 & 6 \\ 1 & & 4 & 7 \\ & & 5 & \\ 2 & & & \end{array}, \qquad Der^-\left(\begin{array}{cccc} & & \square & \square \\ \square & & \square & \square \\ & & \square & \\ \square & & & \end{array}\right) = \begin{array}{cccc} & & 3^- & 4^- \\ 1^- & & 3^- & 4^- \\ & & 3^- & \\ 1^- & & & \end{array}$$

DEFINITION 3.2. Suppose S and T are tableaux of the same shape. Let the word s_i be the ith row of S and let T_i be the ith row of T. Define $[S \mid T] = \prod_i \mathrm{Tab}(\mathrm{s}_i \mid \mathrm{T}_i)$. Hence $[S \mid T] = \prod_i [\mathrm{s}_i \mid \mathrm{T}_i]$.

Now suppose that T is a tableau of shape D. Suppose that $\mathscr{L}$ contains the set of letters present in T and that $\mathscr{P}^-$ contains the indices for all columns present in D. Define an element $[T] \in Super([\mathscr{L} \mid \mathscr{P}])$ indexed by T by

$$[T] = [T \mid Der^-(D)].$$

EXAMPLE 3.2. Let $\mathscr{L} = \mathscr{L}^- = \{a, b, c, d, e, f, g\}$ and let $\mathscr{P} = \mathscr{P}^- = \{1, 2, 3, 4\}$. Let

$$T = \begin{array}{cccc} & & a & d \\ b & & c & e \\ & & f & \\ g & & & \end{array}.$$

Then

$$[T] = \left[\begin{array}{cc|cc} & a\,d & & 3\,4 \\ b & c\,e & 1 & 3\,4 \\ & f & & 3 \\ g & & 1 & \end{array}\right].$$

In other words, $[T]$ is a scalar multiple of $(ad|34)(bce|134)(f|3)(g|1)$. The scalar in this example being 1, we have

$$[T] = \det\begin{pmatrix} (d|3) & (d|4) \\ (a|3) & (a|4) \end{pmatrix} \det\begin{pmatrix} (e|1) & (e|3) & (e|4) \\ (c|1) & (c|3) & (c|4) \\ (b|1) & (b|3) & (b|4) \end{pmatrix} (f|3)(g|1).$$

Definition 3.3. Suppose that D is a shape. Define the *super-Schur module*

$$\mathscr{S}^D(\mathscr{L}) = \mathrm{span}_{\mathbf{Z}}\{[T] : \mathrm{shape}(T) = D \text{ and } T \text{ is filled with letters from } \mathscr{L}\}.$$

In the case that $\mathscr{L}$ is negative (respectively positive) then $\mathscr{S}^D(\mathscr{L})$ is called the Schur (respectively Weyl) module associated with the diagram D. Subsequent to [ABW82], the term Weyl module was introduced (apparently by Boffi) for the co-Schur functors applied to free modules. These terms are justified by the following result. A proof may be found in [T97a].

Proposition 3.4. *Let R be a commutative ring. Let F be a free R-module of rank n. Let α be the $0/1$-matrix having 1's precisely where D has cells.*

If $\mathscr{L} = \mathscr{L}^-$ has cardinality n, then $R \otimes_{\mathbf{Z}} \mathscr{S}^D(\mathscr{L}^-) = L_\alpha(F)$, where L_α is the Akin–Buchsbaum–Weyman "Schur functor" associated to the generalized shape matrix α.

If $\mathscr{L} = \mathscr{L}^+$ has cardinality n, then $R \otimes_{\mathbf{Z}} \mathscr{S}^D(\mathscr{L}^+) = K_\alpha(F)$, where K_α is the Akin–Buchsbaum–Weyman "co-Schur" functor."

Example 3.3. The Weyl module of shape $\begin{smallmatrix}\square&\square&\square\\ &\square& \end{smallmatrix}$ on positive letters a, b is spanned by

$$\begin{bmatrix} a\,a\,a \\ b \end{bmatrix}, \begin{bmatrix} a\,a\,b \\ b \end{bmatrix}, \begin{bmatrix} b\,b\,b \\ a \end{bmatrix}, \begin{bmatrix} a\,b\,b \\ a \end{bmatrix}, \begin{bmatrix} a\,b\,b \\ b \end{bmatrix}, \begin{bmatrix} a\,a\,b \\ a \end{bmatrix}, \begin{bmatrix} a\,a\,a \\ a \end{bmatrix}, \begin{bmatrix} b\,b\,b \\ b \end{bmatrix}$$

in the exterior algebra generated by the anti-commuting variables $(a|1)$, $(a|2)$, $(a|3)$, $(b|1)$, $(b|2)$, $(b|3)$. The last two of the above skew-polynomials are identically 0. Since $\mathscr{S}^{\begin{smallmatrix}\square&\square&\square\\ &\square& \end{smallmatrix}}(\{a, b\})$ is isomorphic to $\mathscr{S}^{\begin{smallmatrix}\square&\square&\square\\ \square& & \end{smallmatrix}}(\{a, b\})$ as a GL_2-module, it has rank 3. In the next section we single out the first three elements as a basis.

4. ROW-CONVEX DIAGRAMS AND STRAIGHT TABLEAUX

The usual bases for skew Weyl modules are indexed by the semistandard Young tableaux, namely all tableaux which weakly increase in their rows and strictly increase in their columns. Example 3.3 showed that this is not the case for more general shapes. Nevertheless, the basis of [ABW82] for skew Weyl modules indexed by standard Young tableaux has a number of properties we wish to preserve. In particular:

(1) The tableaux indexing the basis for $\mathscr{S}^D$ have shape D.

(2) The rows of the tableaux in the indexing set weakly increase.

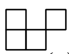
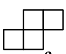
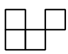
(3) Knowing the number of times a letter appears in each column of a tableau in the indexing set determines that tableau.

(4) It is combinatorially "obvious" when a tableau is in the indexing set.

(5) The elements $[T]$ where T is in the index set form a basis for the module.

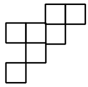
(6) There is an easy to describe algorithm for rewriting $[T]$ in terms of basis elements.

Property (3) underlies the SAGBI-basis algorithms of [Stu93]; in [W94] Woodcock shows that there must exist bases satisfying this property when D is "almost-skew."

Only slightly more complicated shapes,  for instance, fail to simultaneously possess properties (1), (2), (3), and (5). To see this, examine the Specht module associated to this shape. Recall that this is the subspace of the associated Schur module spanned by all tableaux containing letters 1^-, 2^-, 3^-, 4^- with no repeats; here the indexing shape is transposed from the indexing shape used in [Sa91]. This Specht module is isomorphic to the one indexed by  hence has dimension 5. However, one can choose at most 4 tableaux of shape  such that the chosen tableaux satisfy conditions (2) and (3).

We define a class of "straight" tableaux satisfying the above properties. The elements $[T]$ where T is straight and of shape D will form a basis for the super-Schur module $\mathscr{S}^D$ for any "row-convex" shape D.

DEFINITION 4.1. A *row-convex shape*, such as

,

is a shape with no gaps in any row; i.e., if cells (r, i) and (r, k) are in a shape D, then (r, j) is in D, for all $i < j < k$. Since the constructions of Section 3 are not sensitive to the order of rows in a diagram, we assume that *the rows of a row convex diagram are sorted so that higher rows end at least as far to the right as lower rows.*

We can denote any row-convex shape by $\lambda/\underline{m}$ where λ is a partition and $\underline{m}$ is a composition satisfying $m_i \leq \lambda_i$ for all i; a cell is in position (i, j) of $\lambda/\underline{m}$ iff $m_i \leq j \leq \lambda_i$.

Following [GRS87], and employing the notation for inequalities introduced in Section 2, a tableau T with entries in a signed set is *standard* when it $(\lessdot^{+})$-increases across rows and $(\lessdot^{-})$-increases down columns.

I introduce the notion of a *straight tableau* of row-convex shape by slightly relaxing the usual conditions for standardness of a tableau.

DEFINITION 4.2. A row-convex tableau is called *straight* when

(1) The contents of any row $\lessdot^{+}$-increase from left to right, and

(2) Given two cells in the same column, say (i, k) and (j, k) for $i < j$, the entry in the top cell, (i, k), may be $(\gtrdot^{+})$-larger than the entry in (j, k) (i.e., the cells form an *inversion*) only if cell $(i, k - 1)$ exists and its content is $(\gtrdot)$-larger than the content of (j, k).

This definition amounts to requiring that the columns are as close as possible to $(\lessdot^{-})$-increasing, subject to the condition that the rows remain $(\lessdot^{+})$-increasing. A more precise version of the preceding fact is implicit in the correctness of Algorithm Straight-Filling in Fig. 1. A tableau satisfying condition (1) is called *row-standard* and an inversion violating condition (2) is called a *flippable* inversion.

Input: A word w' of length n, and an n-celled row-convex shape D.
Output: A straight tableau T with $w'_T = w'$ or "IMPOSSIBLE" if no such tableau exists.

> Let c_j be the column index of the j in $F(D)$.
> **if** $w'_i \gtrdot^{+} w'_j$ and $c_i = c_j$ for some pair $i < j$, **then return** "IMPOSSIBLE"
> Let T be an empty tableau of shape D
> **for** $k = 1 \ldots n$
>> Let i be the smallest (northmost) index such that $(i, c_k) \in D$ is still empty and either there is no cell in position $(i, c_k - 1)$ or $T_{i,c_k-1} \lessdot^{+} w'_k$.
>> **if** there is no such i **then return** "IMPOSSIBLE"
>>> **else** $T_{i,c_k} \leftarrow w'_k$.

FIG. 1. Algorithm Straight-Filling.

PROPOSITION 4.3. *A skew tableau, T is straight iff it is standard.*

Proof. Since a standard tableau has no inversions, it suffices to prove the only-if part. We prove the contrapositive. We can assume that T is row-standard. Suppose that the cells (i, k), (j, k) with $i < j$ are an inversion. Let k_0 be the least (leftmost) column such that (i, k_0), (j, k_0) is an inversion. If $(i, k_0 - 1)$ exists then by skewness so does $(j, k_0 - 1)$ and thus by assumption $T_{i,k_0-1} <- T_{j,k_0-1} <+ T_{j,k_0}$ hence T is not straight. ∎

COROLLARY 4.4. *The straight tableaux of skew shape with only positively signed letters are the usual semistandard Young tableaux.*

DEFINITION 4.5. Given a tableau T its *column word*, c_T, is the word formed by reading the entries of T from bottom to top in each column, starting with the leftmost column and working towards the right. Its *modified column word* is the word w_T formed by writing the entries of the first column in weakly decreasing order followed by the entries of the second column in weakly decreasing order, etc.

We shall also occasionally require a *reverse column word* w'_T of T formed by writing the entries of the first column of T in weakly increasing order then those of the second column in weakly increasing order, etc.

THEOREM 4.6. *If T and T' are straight tableaux of the same shape, then $T \neq T'$ implies $w_T \neq w_{T'}$. More strongly, if there exists a straight tableau T of shape D with $w_T = w$ then the Algorithm Straight-Filling in Fig.* 1 *produces it.*

Proof. A tableau, T, produced by this algorithm must be straight. If in a fixed column, k, the letter y is inserted into row i by the algorithm while $x <+ y$ was inserted into row $j > i$, then it must be that $T_{i,k-1} -> x$ else the cell $T_{i,k}$ would have been available to x hence x would have been placed there.

Now suppose that the algorithm produces a tableau T with reverse column word w'. Let $\underline{c}$ be as in the algorithm. Any tableau with reverse column word w' can be produced by a similar filling process. Define $\underline{i}$ so that reading through w' and inserting w'_k into cell (i_k, c_k) gives the desired tableau. Let us assume that if w'_k appears in multiple cells in column c_k that the first w'_k in w' is used to fill the northmost appearance in the column, the second is used to fill the second northmost appearance, etc.

Let $\underline{i}$ be the filling sequence corresponding to T; this is the sequence produced by the Algorithm Straight-Filling. Let $\underline{i}'$ be the filling sequence corresponding to some other tableau T'. Let k_0 be the smallest integer such that $i_{k_0} \neq i'_{k_0}$. So in filling T', we have placed w'_{k_0} into cell (i'_{k_0}, c_{k_0}) when acccording to Algorithm Straight-Filling, it could have been put into (i_{k_0}, c_{k_0}) where $i_{k_0} < i'_{k_0}$. By necessity, in filling T', something $(\geq)$-larger than w_{k_0} must be placed in (i_{k_0}, c_{k_0}). By our assumptions about repeated letters in the definition of $\underline{i}$, this inequality is strict. But these facts guarantee that the inversion $\{(i_{k_0}, c_{k_0}), (i'_{k_0}, c_{k_0})\}$ of T' violates condition (2) in the definition of straight tableaux.

The above argument says that if we try to create a straight tableau T with $w'_T = w'$ by reading across w' and sequentially filling its letters into a tableau then at each step the choice of where to insert the letters is forced on us. If at any point during execution of the algorithm there is no place to put a letter which preserves row-standardness, then it is in fact not possible to find a straight tableau with the designated column content and shape. This is precisely the circumstance under which "IMPOSSIBLE" is returned.

We conclude that not only does Straight-Filling produce a straight tableau, but any other tableau, T', having the same modified (equivalently reverse) column word is not straight. ∎

COROLLARY 4.7. *The matrix expressing the super-polynomials* $[T]$ *indexed by straight tableaux as* **Z**-*linear combinations of divided powers monomials in the polynomial superalgebra is in echelon form with* ± 1 *at each pivot. Hence the straight basis elements are linearly independent.*

We defer the proof in order to develop the appropriate orders on basis elements and monomials. Monomials are ordered according to a generalization of the "diagonal term order" in [Stu93] which requires that the smallest monomial in $\det(A)$, where A is a minor of $(x_{i,j})$, be the product of the elements on the diagonal. For compatibility with lexicographic order in Lemma 4.9 this is backwards from the convention in commutative algebra which has $\prod_i (x_{i,i})$ be the *largest* monomial in $\det(A)$.

DEFINITION 4.8. A diagonal term order on *Super*$([\mathscr{L} \mid \mathscr{P}])$ is

(1) A total order, $\prec$, on monomials in *Super*$([\mathscr{L} \mid \mathscr{P}])$ such that for monomials m, m', n, n', the relations $m \prec m'$ and $n \prec n'$ imply that $mn \prec m'n'$ or $mn = 0$ or $m'n' = 0$.

(2) The smallest monomial in a nonzero biproduct $(i_1, \dots, i_k \mid j_1, \dots, j_k)$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ is $\prod_l (i_l \mid j_l)$.

The *default diagonal term order*, $\prec_{\text{diag}}$, that we utilize is characterized below. We order letterplaces $(i \mid j)$ by $(i \mid j) > (i' \mid j')$ when $j < j'$ or when $j = j'$ and $i > i'$. Let M, $N \in$ *Super*$([\mathscr{L} \mid \mathscr{P}])$ be two nonzero monomials. Suppose $(i \mid j)$ is the largest letterplace appearing to a different power in M and N. Write $N \prec_{\text{diag}} M$ when M is divisible by a higher power of $(i \mid j)$ than is N.

EXAMPLE 4.1. Suppose $\mathscr{L} = \{1^-, 2^-\}$ and $\mathscr{P} = \{a^+, x^-\}$ ordered alphabetically; then $(2 \mid a) > (1 \mid a) > (2 \mid x) > (1 \mid x)$. Further, we have

$$(1|x) < (1|x)^{6} < (2|a)(1|x)^{2}(2|x) < (2|a)(2|x)^{2}(1|x)$$
$$< (2|a)(1|a)(1|x)^{2}.$$

The following lemma is immediate.

LEMMA 4.9. *A normalized monomial $\prod_{l=1}^{k}(i_l \mid j_l) \neq 0$ in Super$([\mathscr{L} \mid \mathscr{P}])$ is a monomial written so that $(i_l \mid j_l) \geq (i_{l+1} \mid j_{l+1})$ in the default diagonal term order. For two normalized monomials, $M = \prod_{l=1}^{k}(i_l \mid j_l)$ and $N = \prod_{l=1}^{k}(i'_l \mid j_l)$ differing only in their letters, $M < N$ in the default diagonal term order iff $i_1, \dots, i_k$ is lexicographically less than $i'_1, \dots, i'_k$.*

DEFINITION 4.10. Let Ψ be the function taking a normalized monomial $\prod_{l=1}^{k}(i_l \mid j_l) \in$ *Super*$([\mathscr{L} \mid \mathscr{P}])$ to $i_1, \dots, i_k$.

Typically, we have $\mathscr{P} = \mathscr{P}^-$, in which case normalized monomials are the sorted monomials introduced earlier.

DEFINITION 4.11. Given $p \in$ *Super*$([\mathscr{L} \mid \mathscr{P}])$ and an order $\prec$ on monomials, define the *initial monomial* $\mathrm{init}_{\prec}(p)$ of p to be the smallest (divided powers) monomial appearing in p.

Sometimes the phrase "initial term" will be used when the coefficient of the initial monomial is to be included.

The following result says that in most cases the modified column word of T can be read directly from the smallest monomial appearing in $[T]$.

PROPOSITION 4.12. *If T is a tableau whose rows $(<+)$-increase and whose columns contain no repeated positive letters, then*

$$w_T = \Psi\Big(\mathrm{init}_{\prec_{\mathrm{diag}}}([T])\Big).$$

Proof. Suppose $[T] = \prod_i \mathrm{Tab}(w_{i,c_{i,1}}, w_{i,c_{i,2}}, \dots, w_{i,c_{i,l_i}} \mid c_{i,1}, c_{i,2}, \dots, c_{i,l_i})$. The initial term (with coefficient) of the ith multiplicand is $\prod_j (w_{i,c_{i,j}} \mid c_{i,j})$ and since positive letters never repeat in a column the product of these initial terms is nonzero and hence equals $\mathrm{init}_{\prec_{\mathrm{diag}}}([T])$. ∎

Note that the initial term $\prod_i \prod_j (w_{i,c_{i,j}} \mid c_{i,j})$ appearing above is (up to sign) a basis element in our monomial $\mathbf{Z}$-basis for *Super*$([\mathscr{L} \mid \mathscr{P}])$ and we have proved the following.

PROPOSITION 4.13. *If T is straight of shape D, and if c_l is the index of the column of $F(D)$ containing l then, up to sign*

$$\prod_l ((w'_T)_l \mid c_l) = \operatorname{init}_{\prec_{\text{diag}}}([T]).$$

The coefficient in $[T]$ of its initial monomial is ± 1.

COROLLARY 4.14. *Suppose T is a straight tableau. Then*

$$w_T = \Psi\left(\operatorname{init}_{\prec_{\text{diag}}}([T])\right).$$

We now complete the proof of the independence result.

Proof (*of Corollary* 4.7). Since Theorem 4.6 says that distinct straight tableaux have distinct modified column words, we conclude from Corollary 4.14 that if monomials are ordered by $\prec_{\text{diag}}$ and the polynomials $[T]$ corresponding to straight tableaux are ordered lexicographically by their modified column words, then the matrix expressing the $[T]$ in terms of divided powers monomials is in echelon form with ± 1's as pivots. ∎

COROLLARY 4.15. *Suppose $\sum_i \alpha_i T_i$ is a linear combination of row-standard tableaux such that $\sum_i \alpha_i [T_i] = \sum_j \beta_j [S_j]$ where the S_j are distinct straight tableaux and where all tableaux have the same row-convex shape D. The smallest modified column word of a tableaux in the S_j's is weakly larger* (*lexicographically*) *than the smallest modified column word appearing in the T_i's.*

Proof. Let $p = \sum_i \alpha_i [T_i]$. Let c_l be the column of $F(D)$ containing l. Suppose that $w_{T_{i_0}} \leq w_{T_i}$ for all i and suppose $w_{S_{j_0}} < w_{S_j}$ for all $j \neq j_0$—recall by Theorem 4.6 that distinct straight tableaux have distinct modified column words. We want to show $w_{T_{i_0}} \leq w_{S_{j_0}}$. Now because straight tableaux have distinct modified column words, $\prod_l (w_{S_{j_0} l} \mid c_l)$ is the smallest monomial occurring in p. That means that it must appear in $\sum_i \alpha_i [T_i]$ if that expression is expanded out to a polynomial in $Super([\mathcal{L} \mid \mathcal{P}])$. But if w_{T_i} is always larger than $w_{S_{j_0}}$ then no monomial as small as $\prod_l (w_{S_{j_0} l} \mid c_l)$ can appear in $\sum_i \alpha_i [T_i]$. ∎

The next section shows any $\sum_i \alpha_i [T_i]$ can be rewritten in the above fashion.

5. A STRAIGHTENING ALGORITHM

We produce an explicit two-rowed straightening law for reducing any tableau to a linear combination of straight tableau. This algorithm, Straighten-Tableau shown in Fig. 2, starts with a tableau T and returns a

Input: A row-convex tableau T.
Output: $\sum_\iota \alpha_\iota S_\iota$ such that $[T] = \sum_\iota \alpha_\iota [S_\iota]$ where each S_ι is a straight tableau and $\alpha_\iota \in \mathbf{Z}$.

> **if** T is straight **then** output T.
> **else** there exists a flippable inversion in some rows i, j
>> Let $\sum_\kappa \beta_\kappa \cdot \frac{\cdots v_\kappa \cdots}{\cdots \underline{w_\kappa} \cdots}$ be the output of **row-straighten**$\left(\frac{\cdots \mathrm{T}_i \cdots}{\cdots \mathrm{T}_j \cdots}\right)$;
>> Let N_κ be
>> $\left(\#\text{ pos. letters in } \underline{w_\kappa} + \#\text{ pos. letters in } \mathrm{T}_j\right) \cdot \left(\#\text{ pos letters in } \mathrm{T}_{i+1} \cdots \mathrm{T}_{j-1}\right)$.
>> $$\text{Output } \sum_\kappa (-1)^{N_\kappa} \beta_\kappa \cdot \textbf{straighten-tableau} \left(\begin{array}{c} \cdots \mathrm{T}_1 \cdots \\ \cdots\cdots \\ \cdots \mathrm{T}_{i-1} \cdots \\ \cdots v_\kappa \cdots \\ \cdots \mathrm{T}_{i+1} \cdots \\ \cdots\cdots \\ \cdots \mathrm{T}_{j-1} \cdots \\ \cdots w_\kappa \cdots \\ \cdots \mathrm{T}_{j+1} \cdots \\ \cdots\cdots \end{array} \right).$$

FIG. 2. Algorithm Straighten-Tableau. T_i, T_j, etc., are the ith, jth, etc., rows of the tableau T.

formal linear combination $\sum_i \alpha_i S_i$ of straight tableau with integer coefficients such that $[T] = \sum_i \alpha_i [S_i]$. In each step, the algorithm looks for a pair of rows containing a flippable inversion. If these exist, it applies the sub-algorithm Row-Straighten in Fig. 3 to "straighten" these two rows via the Grosshans–Rota–Stein syzygies of Definition 5.1.

We provide an example of the straightening law below.

EXAMPLE 5.1. Let $\mathscr{L} = \mathscr{L}^- = \{1, 2, \ldots, 8\}$. In each step we shall look for a non-straight tableaux T and locate two rows (say r_1 above r_2) in T containing a flippable inversion. In this example we will mark by a $\star$ every cell in row r_1 weakly right of the left most flippable inversion in those rows and every cell in row r_2 that is weakly left of this flippable inversion and weakly right of a cutoff column c_1. The cutoff c_1 indexes the leftmost column of row r_2 such that either $T_{r_1, c_1 - 1}$ does not exist or $T_{r_1, c_1 - 1} <_+ T_{r_2, c_1}$. In this example, c_1 happens to always index the leftmost column in row r_2. We mark the remaining elements in row r_1 by $\bullet$'s.

The Grosshans–Rota–Stein syzygies (proved for the commutative case $\mathscr{L} = \mathscr{L}^-$ in [DRS76]) say that anti-symmetrizing all the $\star$'d elements in T is the same (up to sign) as collecting all the $\star$'d elements into the row r_1, replacing those removed from row r_2 with these $\bullet$'d elements, and anti-symmetrizing the $\bullet$'d elements. We shall repeatedly apply this identity.

Input: A two-rowed row-convex tableau $T = \begin{smallmatrix} v_{m_1} v_{m_1+1} \cdots v_{\lambda_1} \\ w_{m_2} w_{m_2+1} \cdots w_{\lambda_2} \end{smallmatrix}$ which is row-standard but not straight.
Output: $\sum_\kappa \alpha_\kappa \cdot T_\kappa$ such that
Claim 1: $[T] = \sum_\kappa \alpha_\kappa [T_\kappa]$ where $\alpha_\kappa \in \mathbf{Z}$ and
Claim 2: the column word of $\begin{smallmatrix} \cdots v_\kappa \cdots \\ \cdots w_\kappa \cdots \end{smallmatrix}$ is lexicographically larger than the column word of T.

Let c_2 be the index of the column containing the leftmost flippable inversion.
Let c_1 be the smallest column such that $c_1 \geq m_2$ and either $v_{c_1-1} \not\prec w_{c_1}$ or $c_1 - 1 < m_1$ (i.e. v_{c_1-1} does not exist.)
Let c_3 be the rightmost column such that $w_{c_2} = w_{c_3}$.
if $c_1 < c_2$ **then**
  Let $\sum_{\iota \in I} \beta_\iota T_\iota = Syz_{c_2, c_2+1, \cdots, \lambda_1; c_1, c_1+1, \cdots, c_3}(T)$.
  Expansion $\leftarrow 0$.
  for $\iota \in I$
    if $w_{T_\iota} > w_T$ **then** *Expansion* $\leftarrow$ *Expansion* $+ \beta_\iota T_\iota$.
     else *Expansion* $\leftarrow$ *Expansion* $+ \beta_\iota \cdot$ **row-straighten**(T_ι).
  Output *Expansion*.
else ▷ Comment: $c_1 = c_2$.
  Let c_0 be the leftmost column such that $v_{c_0} \not\succ w_{c_2}$.
▷ Comment: $c_0 < c_2 \Rightarrow v_{c_0} = w_{c_2}$.
  Let $\sum_{\iota \in I} \beta_\iota T_\iota = Syz_{c_0, c_0+1, \cdots, \lambda_1; c_1, c_1+1, \cdots, c_3}(T)$.
  Expansion $\leftarrow 0$.
  for $\iota \in I$
    if $w_{T_\iota} > w_T$ **then** *Expansion* $\leftarrow$ *Expansion* $+ \beta_\iota T_\iota$.
     else *Expansion* $\leftarrow$ *Expansion* $+ \beta_\iota \cdot$ **row-straighten**(T_ι).
  Output *Expansion*.

FIG. 3. Algorithm Row-Straighten. If $\mathscr{L} = \mathscr{L}^-$, then we will always have $w_{T_\iota} > w_T$ so the algorithm will never recurse and instead could have directly output the expressions $Syz(T)$. The expression $Syz(T)$ is defined in Definition 5.1.

So, observing that the entries 4 and 2 form a flippable inversion, we first have

$$\begin{bmatrix} & & 4^\star & 5^\star \\ 1 & 3 & 5 & 7 \\ & & 2^\star & \\ & 3 & 8 & \end{bmatrix} = \begin{bmatrix} & & 2^\star & 5^\star \\ 1 & 3 & 5 & 7 \\ & & 4^\star & \\ & 3 & 8 & \end{bmatrix} - \begin{bmatrix} & & 2^\star & 4^\star \\ 1 & 3 & 5 & 7 \\ & & 5^\star & \\ & 3 & 8 & \end{bmatrix}.$$

But the cells in column 3 and rows 2 and 3 of the first tableau on the right hand side now contain a flippable inversion. We straighten as

$$\begin{bmatrix} & & 2 & 5 \\ 1^{\bullet} & 3^{\bullet} & 5^{\star} & 7^{\star} \\ & & 4^{\star} & \\ & 3 & 8 & \end{bmatrix} = \begin{bmatrix} & & 2 & 5 \\ 1^{\bullet} & 3^{\bullet} & 4^{\star} & 7^{\star} \\ & & 5^{\star} & \\ & 3 & 8 & \end{bmatrix} - \begin{bmatrix} & & 2 & 5 \\ 1^{\bullet} & 3^{\bullet} & 4^{\star} & 5^{\star} \\ & & 7^{\star} & \\ & 3 & 8 & \end{bmatrix}$$
$$+ \begin{bmatrix} & & 2 & 5 \\ 1^{\bullet} & 4^{\star} & 5^{\star} & 7^{\star} \\ & & 3^{\bullet} & \\ & 3 & 8 & \end{bmatrix} - \begin{bmatrix} & & 2 & 5 \\ 3^{\bullet} & 4^{\star} & 5^{\star} & 7^{\star} \\ & & 1^{\bullet} & \\ & 3 & 8 & \end{bmatrix}.$$

Now the first two tableaux above are straight, but the last two are not. We straighten the next to last tableau by

$$\begin{bmatrix} & & 2 & 5 \\ 1^{\bullet} & 4^{\star} & 5^{\star} & 7^{\star} \\ & & 3 & \\ & 3^{\star} & 8 & \end{bmatrix} = \begin{bmatrix} & & 2 & 5 \\ 1^{\bullet} & 3^{\star} & 5^{\star} & 7^{\star} \\ & & 3 & \\ & 4^{\star} & 8 & \end{bmatrix} - \begin{bmatrix} & & 2 & 5 \\ 1^{\bullet} & 3^{\star} & 4^{\star} & 7^{\star} \\ & & 3 & \\ & 5^{\star} & 8 & \end{bmatrix}$$
$$+ \begin{bmatrix} & & 2 & 5 \\ 1^{\bullet} & 3^{\star} & 4^{\star} & 5^{\star} \\ & & 3 & \\ & 7^{\star} & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 3^{\star} & 4^{\star} & 5^{\star} & 7^{\star} \\ & & 3 & \\ & 1^{\bullet} & 8 & \end{bmatrix}$$

and the last tableau by

$$-\begin{bmatrix} & & 2^{\star} & 5^{\star} \\ 3 & 4 & 5 & 7 \\ & & 1^{\star} & \\ & 3 & 8 & \end{bmatrix} = -\begin{bmatrix} & & 1^{\star} & 5^{\star} \\ 3 & 4 & 5 & 7 \\ & & 2^{\star} & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & & 1^{\star} & 2^{\star} \\ 3 & 4 & 5 & 7 \\ & & 5^{\star} & \\ & 3 & 8 & \end{bmatrix},$$

so

$$\begin{bmatrix} & & 4 & 5 \\ 1 & 3 & 5 & 7 \\ & & 2 & \\ & 3 & 8 & \end{bmatrix} = -\begin{bmatrix} & & 2 & 4 \\ 1 & 3 & 5 & 7 \\ & & 5 & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 4 & 7 \\ & & 5 & \\ & 3 & 8 & \end{bmatrix} - \begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 4 & 5 \\ & & 7 & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 5 & 7 \\ & & 3 & \\ & 4 & 8 & \end{bmatrix}$$
$$-\begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 4 & 7 \\ & & 3 & \\ & 5 & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 1 & 3 & 4 & 5 \\ & & 3 & \\ & 7 & 8 & \end{bmatrix} + \begin{bmatrix} & & 2 & 5 \\ 3 & 4 & 5 & 7 \\ & & 3 & \\ & 1 & 8 & \end{bmatrix} - \begin{bmatrix} & & 1 & 5 \\ 3 & 4 & 5 & 7 \\ & & 2 & \\ & 3 & 8 & \end{bmatrix} + \begin{bmatrix} & & 1 & 2 \\ 3 & 4 & 5 & 7 \\ & & 5 & \\ & 3 & 8 & \end{bmatrix}.$$

The first step in verifying Algorithm Straighten-Tableau is to prove that when T is replaced with $\sum_i \beta_i T_i$ by Algorithm Row-Straighten we have $[T] = \sum_i \beta_i [T_i]$. The second step involves showing that each T_i is somehow closer to being straight than was T. The first of these facts is an immediate consequence of the correctness of Algorithm Row-Straighten. This will come down to verifying the identities used in the preceding example. The second will follow from the correctness of Row-Straighten and the fact, proved in Proposition 5.4, that given a tableau T and another tableau T' differing ony in two rows i, j, then the column word of the two-rowed subtableaux consisting of rows i, j of T is less than the corresponding column word determined by T' iff $c_T < c_{T'}$.

The proof of Algorithm Straighten-Tableau thus depends solely on the correctness of Algorithm Row-Straighten. We will prove both claimed properties of Algorithm Row-Straighten for each of the two cases appearing in the algorithm. First we will produce the "determinantal" identities that will be used in Algorithm Row-Straighten.

Define a *shuffle* of a word $\underline{w} = w_1, \ldots, w_n$ into parts of length k, $k' = n - k$ to be an ordered pair of words $\underline{w}'$ and $\underline{w}''$ of $\underline{w}$ having lengths k and k', respectively, such that $\underline{w}'$ and $\underline{w}''$ can be found as a pair of disjoint subwords of $\underline{w}$. Neither $\underline{w}'$ nor $\underline{w}''$ need be contiguous as a subword of $\underline{w}$. When $\underline{w} = 1, \ldots, n$ a shuffle amounts to a permutation σ of the index set $1, \ldots, n$ such that $\sigma_1 < \cdots < \sigma_k$ and $\sigma_{k+1} < \cdots < \sigma_{k+k'}$. Generalizing the length of a permutation we define the *shuffle signature*, $\operatorname{sign}(\underline{\omega}_1, \ldots, \omega_k)$, of a word $\underline{\omega}$ to be the number of pairs $1 \le i < j \le k$ such that $\omega_i > \omega_j$ and $|\omega_i| = |\omega_j| = 1$.

DEFINITION 5.1. Let i, j, k, l be nonnegative integers. Fix a two-rowed row-convex shape D by specifying the starting and ending columns, 1 through $i + j + l$ and m through $m + l + k - 1$ of the top and bottom rows, respectively. For convenience we have let the leftmost column index of the top row be 1, but the column indices can of course be shifted left or right by any integer. With the above convention, we could have $m \le 0$, this produces a skew shape.

Let $T = \begin{smallmatrix} \cdots & \underline{v} & \cdots \\ \cdots & \underline{w} & \cdots \end{smallmatrix}$ be a two-rowed row-convex tableau of shape D. Fix two sequences of column indices, $c_1 < \cdots < c_j$ starting at column $i + l + 1$ and ending at $i + l + j$ and $m \le c'_1 < \cdots < c'_l \le k + l$ starting at column c_1 and ending at $c_1 + l - 1$.

Define $Syz_{c_1, \ldots, c_j; c'_1, \ldots, c'_l}(T)$ to be the formal linear combination

$$\sum_{\substack{\text{all nontrivial shuffles} \\ y_{\sigma(1)} \cdots y_{\sigma(l)};\, y_{\sigma(l+1)} \cdots y_{\sigma(l+j)} \\ \text{of } w_{\sigma(c'_1)} \cdots w_{\sigma(c'_l)} v_{\sigma(c_1)} \cdots v_{\sigma(c_j)}}} -\frac{\alpha_\sigma}{\alpha_e} \cdot T'_\sigma + \sum_{\substack{\text{all shuffles} \\ x_{\tau(1)} \cdots x_{\tau(i)};\, x_{\tau(i+1)} \cdots x_{\tau(i+l)} \\ \text{of } v_1 v_2 \cdots v_{c_1 - 1}}} \frac{\beta_\tau}{\alpha_e} \cdot T''_\tau, \quad (2)$$

where T'_σ, T''_τ, α_σ, and β_τ are defined as follows.

First, define words

$$\underline{x} = v_1, \ldots, v_{c_1-1}, \qquad u_1, \ldots, u_{j+l} = w_{c_1'}, \ldots, w_{c_1'}, v_{c_1}, \ldots, v_{c_j},$$

$$\underline{x_\tau'} = v_{\tau(1)}, \ldots, v_{\tau(i)}, \qquad \underline{u_\sigma'} = u_{\sigma(1)}, \ldots, u_{\sigma(l)},$$

$$\underline{x_\tau''} = v_{\tau(i+1)}, \ldots, v_{\tau(i+l)}, \qquad \underline{u_\sigma''} = u_{\sigma(l+1)}, \ldots, u_{\sigma(l+j)},$$

$$\underline{z_1} = w_m, \ldots, w_{c_1'-1} \qquad \underline{z_2} = w_{c_l'+1}, \ldots, w_{k+l-1}$$

and

$$\underline{z} = \underline{z_1}, \underline{z_2}.$$

Define the tableau T_σ' to be the tableau obtained by sorting the rows of

$$\begin{array}{ccc} \cdots\cdots\ \underline{x}\ \cdots\cdots & & \cdots\cdots\ \underline{u_\sigma''}\ \cdots\cdots \\ \cdots\ \underline{z_1}\ \cdots & \cdots\ \underline{u_\sigma'}\ \cdots & \cdots\ \underline{z_2}\ \cdots \end{array}$$

and T_τ'' to be to be the result of sorting the rows of

$$\begin{array}{cc} \cdots\cdots\cdots\ \underline{u}\ \cdots\cdots\cdots & \cdots\ \underline{x_\tau'}\ \cdots\cdots \\ \cdots\cdots\ \underline{x_\tau''}\ \cdots & \cdots\cdots\ \underline{z}\ \cdots\cdots\ . \end{array}$$

We define

$$\alpha_\sigma = (-1)^{N_1(\tau)} \cdot \frac{\mathbf{c}(\underline{x}, \underline{u_\sigma''})!}{\mathbf{c}(\underline{x})! \cdot \mathbf{c}(\underline{u_\sigma''})!} \cdot \frac{\mathbf{c}(\underline{z_1}, \underline{u_\sigma'}, \underline{z_2})!}{\mathbf{c}(\underline{z_1}, \underline{z_2})! \cdot \mathbf{c}(\underline{u_\sigma'})!}$$

with $N_1(\tau) = |\underline{z_1}| \cdot |\underline{u_\sigma'}| + |\underline{z}| \cdot (i+j+l) + |\underline{u_\sigma'}| \cdot (i+j+l) + \operatorname{sign}(\underline{u_\sigma''}, \underline{u_\sigma'}) + m(\underline{x}, \underline{u_\sigma''}) + m(\underline{z_1}, \underline{u_\sigma'}, \underline{z_2})$ where $m(\underline{\omega}) = \operatorname{sign}(\underline{\omega}) + \binom{\#\text{ neg. letters in }\underline{\omega}}{2}$ and we define

$$\beta_\tau = (-1)^l \cdot (-1)^{N_2(\tau)} \cdot \frac{\mathbf{c}(\underline{u}, \underline{x_\tau'})!}{\mathbf{c}(\underline{u})! \cdot \mathbf{c}(\underline{x_\tau'})!} \cdot \frac{\mathbf{c}(\underline{x_\tau''}, \underline{z})!}{\mathbf{c}(\underline{x_\tau''})! \cdot \mathbf{c}(\underline{z})!}$$

with $N_2(\tau) = |\underline{z}| \cdot (i+j+l) + |\underline{x_\tau''}| \cdot ((i+j+l) - |\underline{u}|) + \operatorname{sign}(\underline{x_\tau'}, \underline{x_\tau''}) + m(\underline{u}, \underline{x_\tau'}) + m(\underline{x_\tau''}, \underline{z})$.

Call the cells in columns $c_1, \ldots, c_j$ of the top row and the cells in columns $c_1', \ldots, c_l'$ of the bottom row of T *marked cells*. If no positive letter appears in both marked and unmarked cells in the top row of T and no positive letter appears in both marked and unmarked cells in the bottom row of T then $\alpha_e = \pm 1$ so the identity holds over $\mathbf{Z}$.

PROPOSITION 5.2. *Let T be a tableau of two-rowed, row-convex shape D whose top row contains its bottom row. Without loss of generality assume the* *top row has leftmost column index* 1. *If column indices* $c_1, \dots, c_j$ *and* $c'_1, \dots, c'_l$ *are chosen as in Definition* 5.1, *then*

$$[T] = \sum_S a_S \cdot [S], \tag{3}$$

where $\sum_S a_S \cdot S = Syz_{c_1, \dots, c_j; c'_1, \dots, c'_l}(T)$ *and both sums are over all* (*w.l.o.g. row-standard*) *tableau of shape D.*

The identity (3) *also holds when D is a two-rowed skew shape* (*i.e.*, $m \le 0$ *in Definition* 5.1) *and* $l + j$ *exceeds the number of columns in D. In this case, the right-hand summand in expression* (2) *vanishes.*

This result follows from the more general Theorem 10 of [GRS87] but for completeness, we sketch a proof relying on positive letters and the polarization operators of Section 8 that bypasses the Hopf algebra techniques of [GRS87]. The proof provides a much simpler though less explicit definition of the expressions $Syz_{\underline{c}, \underline{c}'}$.

Proof. First, we prove the following proposition directly by checking that the monomials arising from the expansion of each expression have the same coefficients. Details may be found in [T97a].

Proposition 5.3. *Let a, b, c be positive letters. Let i, j, k, l be nonnegative integers. Let* $1, 2, \dots, i + j + l$ *be negative letters. Fix a two-rowed row-convex shape D whose top row contains its bottom row by specifying the starting and ending columns,* 1 *through* $i + j + 1$ *and m through* $m + l + k - 1$ *of the top and bottom rows, respectively. The following identity holds for tableaux of shape D,*

$$\left(\begin{matrix} a^{(i+l)} \ b^{(j)} \\ b^{(l)} \ c^{(k)} \end{matrix} \middle| \begin{matrix} 1 \ 2 \ \cdots\cdots\cdots \ i + j + l \\ m \ m+1 \ \cdots \ m + l + k - 1 \end{matrix} \right)$$

$$= (-1)^l \left(\begin{matrix} b^{(j+l)} \ a^{(i)} \\ a^{(l)} \ c^{(k)} \end{matrix} \middle| \begin{matrix} 1 \ 2 \ \cdots\cdots\cdots \ i + j + l \\ m \ m+1 \ \cdots \ m + l + k - 1 \end{matrix} \right), \tag{4}$$

where

$$\left(\begin{matrix} \cdots \mathrm{S}_1 \cdots \\ \cdots \mathrm{S}_2 \cdots \end{matrix} \middle| \begin{matrix} \cdots \mathrm{T}_1 \cdots \\ \cdots \mathrm{T}_2 \cdots \end{matrix} \right) = (\cdots \mathrm{S}_1 \cdots \mid \cdots \mathrm{T}_1 \cdots)(\cdots \mathrm{S}_2 \cdots \mid \cdots \mathrm{T}_2 \cdots)$$

and where $l^{(k)} = l^k / k!$.

Choosing $\underline{v} = v_1, \dots, v_{i+l}$, $\underline{w} = w_1, \dots, w_k$, and $\underline{u} = u_1, \dots, u_{j+l}$, and applying the product,

$$\frac{1}{\mathbf{c}(\underline{v})!\, \mathbf{c}(\underline{w})!\, \mathbf{c}(\underline{u})!} D_{v_1, a} D_{v_2, a} \cdots D_{v_{i+l}, a} D_{w_1, c} \cdots D_{w_k, c} D_{u_1, b} \cdots D_{u_{j+l}, b},$$

of polarizations to the identity in Proposition 5.3 completes the proof when m in Definition 5.1 is positive. For $m \leq 0$, we recover the identity used traditionally to straighten skew tableaux. A proof follows by recognizing that the right-hand side of Eq. (4) vanishes when $j + l > i + j + l - m + 1$.

We have just verified that any formal linear combination of tableau with integer coefficients produced by Algorithm Row-Straighten satisfies Claim 1 made in the algorithm specifications; we now go to work on the heart of the proof, namely Claim 2. In the process of proving Claim 2, we will check that Row-Straighten terminates.

PROPOSITION 5.4. *Let D be the row-convex shape $(\lambda_1, \lambda_2)/(m_1, m_2)$ with $m_1 < m_2$.*

Given a non-straight, row standard, two rowed, row-convex tableau, T, of shape D, Algorithm Row-Straighten produces a formal linear combination of tableaux each of which has a lexicographically larger column word than c_T.

Proof. The proof is by induction on c_T.

Suppose that T is the tableau

$$\begin{array}{lcccccccc} \text{Column \#} & m_1 & m_1 + 1 & m_2 & m_{2+1} & & \lambda_2 & \lambda_1 & \\ & & & & & & & & \\ & v_{m_1} & v_{m_1+1} & & & \cdots & & v_{\lambda_1} & . \\ & & & w_{m_2} & w_{m_2+1} & \cdots & w_{\lambda_2} & & \end{array}$$

We set up a straightening syzygy that expresses T in terms of tableau T_i such that c_T is always lexically smaller than c_{T_i}. We first describe the structure of T with respect to its leftmost flippable inversion. Let q be minimal such that if column $q + 1$ were to contain an inversion, then that inversion would be flippable. Thus $q = \min_{m_2 \leq i \leq \lambda_2, v_{i-1} <_+ w_i} i - 1$. The presence of a flippable inversion guarantees that q exists. The value of c_1 in Algorithm Row-Straighten is $q + 1$. The first column strictly right of $c_1 - 1$ that actually *has* an inversion has index $c_2 = \min_{c_1 \leq j \leq \lambda_2, v_j <_+ w_j} j$. This inversion must be flippable; thus c_2 indexes the column of the leftmost flippable inversion in the tableau.

Two cases arise in the algorithm namely $c_1 < c_2$ and $c_1 = c_2$. The pictures in Figs. 4 and 5 outline these situations. The symbol, $\bullet$, indicates a cell in the diagram. An arrow from one cell to another indicates that the contents of the first cell are larger than the contents of the second. The decoration of an arrow by $-$ (respectively $+$) indicates that the contents of the cells at either end may be equal if these contents are negatively (respectively positively) signed. Sequences of cells surrounded by parentheses or braces may be omitted.

If no positive letter appears multiple times in T, then Cases I and II can be treated simultaneously. We begin with Case I.

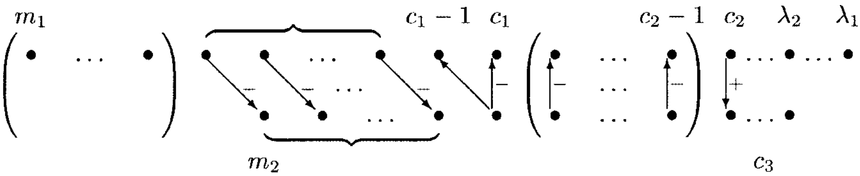


FIG. 4. Case I. $c_1 < c_2$. Relations between entries in a two-row tableau being straightened by Algorithm Row-Straighten. All entries in the bottom row from c_2 through c_3 are equal but distinct from any entry in column $c_3 + 1$ of that row.

Apply Corollary 5.2 to write

$$\begin{bmatrix} v_{m_1} & & \cdots & \cdots & \cdots\ \overline{v_{c_2} \quad \cdots \quad v_{\lambda_1}} \\ & w_{m_2} & & \cdots\ \underline{w_{c_1} \quad \cdots \quad w_{c_2} \quad \cdots \quad w_{c_3}} & \cdots \end{bmatrix} = \underset{c_2,\ldots,\lambda_2;c_1\cdots c_3}{Syz} = B + A, \tag{5}$$

where B (respectively A) is the first (respectively second) summation in the $Syz_{c_2,\ldots,\lambda_2;c_1\cdots c_3}(T)$ as defined in expression (2). The over/underlines are visual aids which indicate the "marked" entries used to defined $Syz(T)$.

It suffices to show that each tableau appearing in A or B has lexically larger column word than c_T.

Suppose T' appears in A. We can write

$$T' = \begin{matrix} \boxed{x_{m_1} \quad \cdots \quad x_{t-1} \quad w_{c_1} \quad \cdots \quad w_{c_3}} & v_{c_2} \quad \cdots \quad \cdots \quad \cdots \quad v_{\lambda_1} \\ \qquad \boxed{w_{m_2} \quad \cdots \quad w_{c_1-1} \quad y_1 \quad \cdots \quad y_{c_2-t}} & w_{c_3+1} \quad \cdots \quad w_{\lambda_2}, \end{matrix}$$

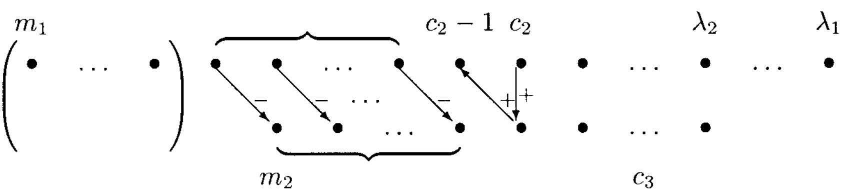


FIG. 5. Case II. $c_1 = c_2$. Relations between entries in a two-row tableau being straightened by Algorithm Row-Straighten. All entries in the bottom row from c_2 through c_3 are equal but distinct from any entry in column $c_3 + 1$ of that row. If $c_0 < c_2$, then the entries in the top row that equal the bottom row entry in column c_2 must start at c_0 and extend at least as far as $c_2 - 1$.

where $x_{m_1} \cdots x_{t-1}$; $y_1 \cdots y_{c_2-t}$ is a shuffle of $v_{m_1} \cdots v_r$, where $t = c_2 + c_1 - c_3 - 1$, and where the boxed entries must be sorted in order to give a row-standard tableau. Denote the entries in the bottom row by $z_{m_2}, \dots, z_{\lambda_2}$ and the entries in the top row by $\varpi_{m_1}, \dots, \varpi_{\lambda_1}$.

To check that only the boxed entries need to be sorted in the top row, it suffices to observe that the x_i are taken from $v_{m_1}, \dots, v_r$ and that $v_{c_2} \not\Rightarrow w_{c_2} = w_{c_3}$. Checking the bottom row, it suffices to note that $w_{c_3+1} \not\Rightarrow v_{c_2-1}$.

Now let $k+1$ index the leftmost column in the top row in which T' differs from T. In fact, $k = \min_{m_1 \le i \le t,\, \varpi_i \ne v_i} i - 1$ which follows from being in Case I: Suppose $\varpi_i = v_i$ for $m_1 \le i < t = c_1 - 1 - c_3 + c_2$. Since by construction $v_{t-1} \lessdot v_{c_1-1} < w_{c_1}$, we have that $x_i = v_i$ for all i as above and thus the boxed elements in the top row are already in order. But then $\varpi_t = w_{c_1} \ne v_t$, since by Case I, $v_{c_1-1} < w_{c_1}$. So $k < t$.

Now we examine the column words. Our construction shows $\underline{\varpi} \ge \underline{v}$, so by the preceding paragraph $\underline{\varpi} > \underline{v}$. So if $k+1 < m_2$ we conclude directly that $c_{T'}$ is lexically larger than c_T.

Suppose that $k+1 \ge m_2$. We show that $v_{k+1} \le y_1$. Suppose to the contrary that $v_{k+1} > y_1$. Since y_1 comes from $v_{m_1}, \dots, v_r$ this says that $y_1 = v_j$ for some $j \le k$ and $y_1 \ne v_{j'}$ for $j' > k$. Now the upper row of T' still contains $v_1 \cdots v_k$ even though a y_1 has been removed to the bottom row. But this implies that y_1 also appears in $w_{c_1} \cdots w_{c_2}$ which is impossible since $w_{c_1} \not\Rightarrow v_{c_1-1} \not\Rightarrow v_{k+1} > y_1$.

Thus since $k+1 \le t < c_1$ the diagram for Case I shows that $w_{k+1} < v_{k+1}$, hence $w_{k+1} < y_1$. So, after sorting, we find that $z_{m_2} = w_{m_2}$, $z_{m_2+1} = w_{m_2+1}, \dots, z_{k+1} = w_{k+1}$. So in tableau T and T', the columns $m_1, \dots, k$ agree as does the bottom entry of column $k+1$. But the top entry in column $k+1$ is larger in T' than in T. Hence $c_{T'}$ is lexically larger than c_T.

At last we deal with tableaux appearing in B in Eq. (5). Recall that tableaux in B arise from nontrivially shuffling the over/underlined entries and then resorting the rows. Let $\underline{u} = w_{c_1} \cdots w_{c_3} v_{c_2} \cdots v_{\lambda_1}$. Let $\underline{u}''$, $\underline{u}'$ be a shuffle of $\underline{u}$ into two parts of size $\lambda_1 - c_2 + 1$ and $c_3 - c_1 + 1$, respectively. Since $w_{c_1} \not\Rightarrow v_{c_1-1}$, such a tableau will look like

$$T' = \begin{array}{ccccccccc} v_{m_1} & & \cdots & v_{c_1-1} & \multicolumn{5}{|c|}{v_{c_1} \quad \cdots \quad v_r \qquad \cdots\cdots\ \underline{u}''\ \cdots\cdots} \\ & w_{m_2} & \cdots & w_{c_1-1} & \multicolumn{5}{|c|}{\cdots\cdots\ \underline{u}'\ \cdots\cdots \qquad w_{c_3+1} \quad \cdots \quad w_{\lambda_2}} \end{array},$$

where, as before, the boxed elements must be sorted so that T' will be row standard. Again denote the top and bottom rows of T' by $\varpi_{m_1}, \dots, \varpi_{\lambda_1}$ and $z_{m_2}, \dots, z_{\lambda_2}$.

Now let $k+1$ be the leftmost column in which the bottom rows of T and T' disagree. We claim $c_1 \le k+1 \le c_3$ and that $z_{k+1} > w_{k+1}$. By construction $\underline{u}' \neq w_{c_1} \cdots w_{c_3}$. Write $\underline{u}' = \underline{u}'_{q+1} \cdots \underline{u}'_{c_3}$. Let j be minimal such that $\underline{u}'_j \neq w_j$. Because $v_{c_2} +\!\!> w_{c_2} = w_{c_3}$, this implies that $\underline{u}'_j > w_j$. But since also $w_{c_3+1} > w_{c_3}$, we find $z_j > w_j$, and $z_i = w_i$ for all $i < j$, so $k + 1 = j$.

Subcase 1. Suppose $k < c_2 - 1$. Any letter appearing in the multiset difference $\underline{u}'' - \{\{v_{c_2}, \dots, v_{\lambda_1}\}\}$ is $(+\!\!>)$-greater than w_{k+1}. But since $k < c_2 - 1$, the picture of Case I shows that $v_{k+1} <\!\!- w_{k+1}$; this means that on resorting the boxed elements of the top row, every element in $\underline{u}'$ stays in column $k + 2$ or higher. Hence columns $m_1 \cdots k$ agree in T and T'. But the bottom of column $k + 1$ is larger in T' than T. Thus $c_{T'}$ is lexically larger than c_T.

The above argument also generates the fact (unused in this proof, but see the comment after Corollary 5.7) that T and T' agree in the top element of column $k + 1$.

Subcase 2. Suppose that $k \geq c_2 - 1$. This says that the bottom rows of T, T' agree at least through column $c_2 - 1$. Since $v_{c_2-1} <\!\!+ w_{c_2}$, we have immediately that the top rows of T, T' agree through column $c_2 - 1$. Now either the bottom of column c_2 changes (hence increases) so $c_{T'}$ is lexically larger than c_T and we are done or the number of positive letters in the bottom row that equal w_{c_2} decreases. In the latter case, not only do the tableaux T, T' agree up to column $c_2 - 1$ but T' still has a flippable inversion in column c_2 since the entry in the top of that column now equals the positive letter w_{c_2} that remains at the bottom.

We repeat the straightening law on T', producing some tableaux with lexicographically larger modified column words and some tableaux that are unchanged in columns smaller than c_2 and unchanged at the bottom of column c_2 but which have fewer copies of w_{c_2} in their bottom rows. Eventually, we must run out of positive letters equal to w_{c_2} in the bottom row and so eventually the modified column word increases.

We now treat Case II (see Fig. 5). Here $c_1 = c_2$.

We replace $[T]$ with $Syz_{c_0, \dots, \lambda_1; c_1, \dots, c_3}$ where c_0 is minimal such that $v_{c_0} +\!\!> w_{c_2}$. As before, Corollary 5.2 lets us write

$$\begin{bmatrix} v_{m_1} & & \cdots & v_{c_0} & \cdots & v_{c_2} & & \cdots & & v_\lambda \\ & w_{m_2} & \cdots & w_{c_0} & \cdots & w_{c_2} & \cdots & w_{c_3} & \cdots & \end{bmatrix} = B + A \qquad (6)$$

just as in Eq. (5). The entries that have been marked twice are positive letters all equal to each other.

Suppose T' appears in A. To this purpose, let $w = v_{m_1} \cdots v_r$ be the word being shuffled. It is easily verified that

$$T' = \begin{array}{cccccccccccc} x_{m_1} & \cdots & x_{t-1} & v_{c_0} = & \cdots & = w_{c_3} & v_{c_2} & \cdots & & \cdots & & v_{\lambda_1} \\ & \boxed{w_{m_2} \quad \cdots \quad \cdots \quad w_{c_2-1} \quad y_1 \quad \cdots \quad y_{c_0-t}} & & & & & & & w_{c_3+1} & \cdots & & w_{\lambda_2}, \end{array}$$

where $t = c_0 - c_3 + c_2 - 1$, $x_{m_1} \cdots x_{t-1}$; $y_1 \cdots y_{c_0-t}$ is a shuffle of $v_{m_1}, \dots, v_{c_0-1}$ and where the boxed entries must be sorted in order to get a row-standard tableau. Maintain the notation $T' = \begin{smallmatrix} \cdots & \varpi & \cdots \\ \cdots & z & \cdots \end{smallmatrix}$.

Since $t < c_0$ we have $v_{c_0} \neq v_t$ and thus $k = \min_{m_1 \leq i \leq t,\, \varpi_i \neq v_i} i - 1$ is well defined. As in Case I, if $k + 1 < m_2$ we conclude directly that $c_{T'}$ is lexically larger than c_T.

Suppose that $k + 1 \geq m_2$. We show that $v_{k+1} \leq y_1$. Suppose to the contrary that $v_{k+1} > y_1$. Since y_1 comes from $v_{m_1}, \dots, v_{c_0-1}$ this says that $y_1 = v_j$ for some $j \leq k$ and $y_1 \neq v_{j'}$ for $j' > k$. But this says that if the letter y_1 occurs in the $y_1 \cdots y_{c_0-t}$ part of the shuffle, then the $x_{m_1} \cdots x_k$ part cannot start with $v_{m_1}, \dots, v_k$—contradiction.

Thus since the diagram for Case II shows that $w_{k+1} < v_{k+1}$, we find $w_{k+1} < y_1$. So, after sorting, we discover that $z_{m_2} = w_{m_2}$; $z_{m_2+1} = w_{m_2+1}$; $\dots$; $z_{k+1} = w_{k+1}$. So in tableaux T, T', the columns $m_1, \dots, k$ agree as does the bottom entry of column $k + 1$. But the top entry in column $k + 1$ is larger in T' than in T. Hence $c_{T'}$ is lexically larger than c_T.

Suppose now that T' appears in B in Eq. (6). Define $c_4 = \min_{c_2 \leq i \leq \lambda_1,\, w_{c_2} < v_i} i$. Since $w_{c_2} \not> v_{c_2-1} = v_{c_0}$, we have

$$\text{Column: } c_4 + s - 1$$

$$T' = \begin{array}{cccccccc} v_{m_1} & & \cdots & v_{c_2-1} w_{c_2} = & \cdots & = w_{c_2} & \cdots\cdots\cdots \; W' \; \cdots\cdots\cdots & \\ & w_{m_2} & \cdots & w_{c_2-1} w_{c_2+s} & \cdots & w_{c_3} & \boxed{\cdots \; W'' \; \cdots \; w_{c_3+1} \; \cdots \; w_{\lambda_2}}, & \end{array}$$

where W', W'' is a shuffle of $v_{c_4} \cdots v_{\lambda_1}$ into parts of size $\lambda_1 - c_4 - s + 1$ and $1 \leq s \leq c_3 - c_2 + 1$, respectively and, as before, the boxed elements must be sorted so that T' will be row standard.

The top rows of T, T' agree through column $c_2 - 1$. Again either the bottom of column c_2 increases and we are done or the number of positive letters in the bottom row that equal w_{c_2} decreases. So iterating the straightening law on T' eventually increases the modified column word. ∎

The Algorithm Row-Straighten specializes to the classical straightening for skew and partition shaped tableau when $m_1 \geq m_2$. The preceding result implies that the column word also increases in the skew case.

COROLLARY 5.5. *Proposition* 5.4 *also holds with* $m_1 \geq m_2$.

Proof. We use what is often called the method of fake letters. Fill cells $m_2 - 1, \ldots, m_1 - 1$ in the top row with new negative letters disjoint from and smaller than the letters in $\mathscr{L}$; we will name these letters f_{m_2-1}, $\ldots, f_{m_1-1}$. Straighten this new tableau.

The tableaux appearing in expression B in the preceding proof have the "fake" letters $f_{m_2-1}, \ldots, f_{m_1-1}$ in the same positions as does the original tableau T. If we apply the algebra homomorphism sending $(f_i \mid j)$ to $\delta_{i,j}$, $[T']$ is sent to 0 for all T' in the expression A in the preceding proof and the fake letters are erased from all other tableaux in the expression. ∎

We have now established the correctness of Algorithm Row-Straighten.

THEOREM 5.6. *The straight tableaux of shape D form a* $\mathbf{Z}$*-basis for* $\mathscr{S}^D$ *and Algorithm Straighten-Tableau expands any generator of* $\mathscr{S}^D$ *in terms of this basis. Further, given a row-standard tableau* T, *the expansion of* $[T]$ *is in terms of tableaux with larger column words than* w_T.

By Corollary 4.15 carefully analyzing the proof Proposition 5.4 we can extend the preceding result.

COROLLARY 5.7. *Algorithms Row-Straighten and Straighten-Tableau produce tableaux with weakly larger modified column words than that of the input tableau.*

6. FLAGGED SUPER-SCHUR MODULES

The *flagged* Schur modules $\mathscr{S}_f^D$ have been the subject of considerably interest (see for instance [LS90, RS96, LM97]). We apply the preceding results to flagged super-Schur modules of row-convex shape. The fact that the initial terms in the straight basis are distinct allow the straight bases to descend to bases of the corresponding flagged module. I will start by formalizing the notion of a flagged super-Schur module.

DEFINITION 6.1. Let $\underline{f}$ be a weakly increasing sequence of letters in the alphabet $\mathscr{L}$. Regard this sequence as indexed by elements of $\mathscr{P}$. The *flagged super-Schur module* $S_{\underline{f}}^D(\mathscr{L})$ is the subquotient of *Super*$([\mathscr{L} \mid \mathscr{P}])$ equal to the image of the submodule $\mathscr{S}^D(\mathscr{L})$ under the map $\phi_{\underline{f}}$ which quotients *Super*$([\mathscr{L} \mid \mathscr{P}])$ by setting $(l \mid p) = 0$ whenever $l > f_p$.

A tableau T is *flagged* if, in each column i, T has no entry exceeding f_i.

If T is row-standard (of any shape) and fails to be flagged, then $\phi_f([T]) = 0$—since each monomial in the expansion of any row in which the flagging condition is violated has some factor $(l \mid p)$ with $l > f_p$.

Classical results (see [Sta76]) tell us that if D is a skew-tableau then a basis for $\mathscr{S}_f^D$ is given by all $[T]$ such that T is standard and flagged. This result carries over to flagged row-convex super-Schur modules.

THEOREM 6.2. *Let D be a row-convex shape. Fix a weakly increasing flag f. A basis for $\mathscr{S}_f^D(\mathscr{L})$ is given by the elements $\phi_f([T])$ where T runs over all flagged, straight tableaux of shape D with entries chosen from $\mathscr{L}$.*

Proof. It suffices to show that the basis elements are linearly independent, indeed that their initial terms under any diagonal term order are still distinct. This follows by Proposition 4.12 and the observation immediately following it since the tableaux T are both straight and flagged. ▮

This result has the following easy generalization. Let f, g both be weakly increasing sequences of letters in $\mathscr{L}$ indexed by elements of $\mathscr{P}$ such that $f \leq g$ componentwise. Define the doubly flagged super-Schur module $\mathscr{S}_{f,g}^D(\mathscr{L})$ to be the image of $\mathscr{S}^D(\mathscr{L})$ under the map $\phi_{f,g}$ quotienting *Super*$([\mathscr{L} \mid \mathscr{P}])$ by the ideal generated by $\{(l \mid p) : l \notin f_l, \ldots, g_l\}$. Call a tableau T *doubly flagged* with respect to f, g if every entry in column i is between f_i and g_i. The same proof as above shows the following.

THEOREM 6.3. *A basis for $\mathscr{S}_{f,g}^D(\mathscr{L})$ is given by the elements $\phi_f([T])$ where T runs over all doubly flagged, straight tableaux of shape D with entries chosen from $\mathscr{L}$.*

When $\mathscr{L} = \mathscr{L}^-$ and D is skew, this result appears in [Sta76].

7. GROEBNER AND SAGBI BASES

The basis theorems developed above have various ring-theoretic applications. For convenience, we state them in terms of commutative rings; i.e., we assume that the alphabet $\mathscr{L}$ consists entirely of negative letters.

DEFINITION 7.1. Let D be a shape. Let R^D be the subalgebra generated by all $[T]$ for all tableaux T of shape D. Allowing that $[T]$ has degree 1, this is a graded algebra generated by its degree 1 part. These algebras turn out to be the homogeneous coordinate rings of certain *configuration varieties* (see [M98], for example) embedded in projective space by a combination of Plucker embeddings, Segre products, and Veronese maps. Configuration varieties parameterize a tuple of subspaces of n-space subject to certain lower bounds on the dimensions in which they may intersect. The results in this section are stated for the rings R^D but they hold equally for the subquotient rings generated by the (doubly) flagged modules $\mathscr{S}^D_{\underline{f},\underline{g}}(\mathscr{L})$.

Below we produce a Groebner basis for R^D. First we establish some notation.

DEFINITION 7.2. Let D be a shape. Let $T_1, \ldots, T_k$ be tableaux of shape D. Define $T_1 \circ T_2 \circ \cdots \circ T_k$ to be the tableau (no longer of shape D) whose $ki + j$th row (with $1 \leq j \leq k$) is the ith row of T_j.

DEFINITION 7.3. Let D be an n-rowed, row-convex shape.

Consider the polynomial ring R whose variables consist of all row-standard tableaux of shape D on $\mathscr{L}$. We define an order on this ring as follows. For two tableaux of shape D, we define $T' < T''$ when $c_{T'} < c_{T''}$. Now consider monomials $\prod_{i=1}^k T_i'$ and $\prod_{j=1}^k T_j''$ satisfying $T_r' \leq T_s'$ and $T_r'' \leq T_s''$ for all $r < s$. We define $\prod_{i=1}^k T_i' < \prod_{j=1}^k T_j''$ when $c_{T_1' \circ \cdots \circ T_k'}$ is smaller in lexicographic order than $c_{T_1'' \circ \cdots \circ T_k''}$.

We collect the following facts about this order.

LEMMA 7.4. *Let R and $<$ be as in Definition* 7.3.

(1) *If $T_1 \circ \cdots \circ T_k$ is straight, then $T_1 \leq \cdots \leq T_k$.*

(2) *If $T_1 \leq \cdots \leq T_k$ then $c_{T_1 \circ \cdots \circ T_k}$ is maximal among $c_{T_{\sigma 1} \circ \cdots \circ T_{\sigma k}}$ for all permutations $\sigma \in S_k$.*

(3) *If $T_1 \leq \cdots \leq T_k$ then $c_{T_1 \circ \cdots \circ T_k} \leq c_{T_1' \circ \cdots \circ T_k'}$ implies $T_1 \cdots T_k \leq T_1' \cdots T_k'$ regardless of whether the tableaux $T_1', \ldots, T_k'$ are in increasing order.*

(4) *The order $<$ is indeed a term order, namely for monomials M, M', $M'' \in R$, with M', M'' of the same degree, $M' < M''$ implies $MM' < MM''$.*

Proof (*Sketch*). Parts (1) and (2) are verified by examining the behavior of the first k elements in the column word and inducting on the number of cells in D. Part (3) is immediate from part (2). Part (4) is a consequence of part (3): Write the tableaux appearing in M', M'' as increasing sequences $\underline{M}'$ and $\underline{M}''$. Insert the tableaux of M into the sequence $\underline{M}'$ to create an increasing sequence $\underline{S}'$. Create a new, not necessarily increasing, sequence $\underline{S}''$ by inserting the tableaux of M into the same relative positions in M'' as were used above to create $\underline{S}'$. But now $c_{S_1' \circ S_2' \circ \cdots} < c_{S_1'' \circ S_2'' \circ \cdots}$, so applying part (3) completes the proof. ∎

DEFINITION 7.5. With R as in Definition 7.3, we write down two classes of polynomials that will compose the desired Groebner basis,

$$T - \sum_{\kappa} \beta_{\kappa} \cdot \begin{array}{c} \cdots \mathrm{T}_1 \cdots \\ \cdots\cdots \\ \cdots \mathrm{T}_{i-1} \cdots \\ \cdots v_{\kappa} \cdots \\ \cdots \mathrm{T}_{i+1} \cdots \\ \cdots\cdots \\ \cdots \mathrm{T}''_{j-1} \cdots \\ \cdots w_{\kappa} \cdots \\ \cdots \mathrm{T}''_{j+1} \cdots \\ \cdots\cdots \\ \cdots \mathrm{T}_n \cdots \end{array} \tag{7}$$

$$T' \cdot T'' - \sum_{\kappa} \beta_{\kappa} \cdot \begin{array}{c} \cdots \mathrm{T}'_1 \cdots \\ \cdots\cdots \\ \cdots \mathrm{T}'_{i-1} \cdots \\ \cdots v_{\kappa} \cdots \\ \\ \cdots \mathrm{T}'_{i+1} \cdots \\ \cdots\cdots \\ \cdots\cdots \\ \cdots \mathrm{T}'_n \cdots \end{array} \cdot \begin{array}{c} \cdots \mathrm{T}''_1 \cdots \\ \cdots\cdots \\ \cdots\cdots \\ \cdots \mathrm{T}''_{j-1} \cdots \\ \\ \cdots w_{\kappa} \cdots \\ \cdots \mathrm{T}''_{j+1} \cdots \\ \cdots\cdots \\ \cdots \mathrm{T}''_n \cdots \end{array} \tag{8}$$

where in the relations (7) we have

$$\sum_{\kappa} \beta_{\kappa} \cdot \begin{array}{c} \cdots \underline{v_{\kappa}} \cdots \\ \cdots \underline{w_{\kappa}} \cdots \end{array} = Syz_{a_1, \ldots, a_r; b_1, \ldots, b_s} \begin{pmatrix} \cdots \mathrm{T}_i \cdots \\ \cdots \mathrm{T}_j \cdots \end{pmatrix}$$

and in (8),

$$\sum_{\kappa} \beta_{\kappa} \cdot \begin{array}{c} \cdots \underline{v_{\kappa}} \cdots \\ \cdots \underline{w_{\kappa}} \cdots \end{array} = Syz_{a_1, \ldots, a_r; b_1, \ldots, b_s} \begin{pmatrix} \cdots \mathrm{T}'_i \cdots \\ \cdots \mathrm{T}''_j \cdots \end{pmatrix}.$$

Before stating the theorem we recall our convention that the initial term of a polynomial is the *smallest* term in the polynomial and define the degree of a Groebner basis to be the highest degree of a polynomial appearing in a Groebner basis.

THEOREM 7.6. *Let D and R be as in Definition* 7.3.
If I is the kernel of the ring map sending a row-standard tableau T to $[T]$, then I has a degree 2 Groebner basis consisting of the straightening relations

(7) *and all polynomials* (8) *where* T, T', T'' *range over all row-standard tableaux of shape* D *on* $\mathscr{L}$.

We allow the indices $a_1, \ldots, a_r$ *and* $b_1, \ldots, b_s$ *appearing in the relations* (7) *and* (8) *to be chosen as follows*: *Let* E *be the two-rowed subshape of* D *consisting of rows* i, j *of* D. *If* E *is not skew then* $\underline{a}$ *and* $\underline{b}$ *are allowed to range over all sequences of consecutive column indices such that* $\underline{a}$ *ends in the last column of* T'_i *and if* E *is skew*, $\underline{a}$ *and* $\underline{b}$ *range over all sequences of column indices such that* $r + s$ *is larger than the number of columns in* E.

Proof. Proposition 5.2 verified that the polynomials in the claimed Groebner basis are in I.

Since the straight tableaux of any fixed shape are linearly independent, it is enough to show that any monomial $T_1 \cdots T_k$ where $T_1 \circ \cdots \circ T_k$ is not straight will be reduced via the Groebner basis. If any of the T_i is not straight, then the fundamental properties of the straightening algorithm imply that one of the relations in (7) provides the desired reduction.

It now suffices to show that if $T := T' \circ T''$ is not straight then there exists a relation in the Groebner basis whose initial term is $T' \cdot T''$. Let T'_i be the ith row of T' and let T''_j be the jth row of T''. Since T is not straight, there exists $i < j$ (or $i > j$) such that the tableau

$$\begin{matrix} \cdots \mathrm{T}'_i \cdots \\ \cdots \mathrm{T}''_j \cdots \end{matrix} \qquad \left(\text{respectively} \ \begin{matrix} \cdots \mathrm{T}''_j \cdots \\ \cdots \mathrm{T}'_i \cdots \end{matrix} \right)$$

is not straight or there exists i such that the tableau

$$\begin{matrix} \cdots \mathrm{T}'_i \cdots \\ \cdots \mathrm{T}''_i \cdots \end{matrix} \ \text{is not straight}.$$

But then Proposition 5.4 and Corollary 5.5 show that choosing $\underline{a}$ and $\underline{b}$ as in Algorithm Row-Straighten gives $T' \cdot T''$ as the initial term of the polynomial (8). ∎

The Groebner basis of Theorem 7.6 is not reduced, nor are the initial terms of its elements necessarily distinct. The initial terms can be made distinct by choosing a unique flippable inversion for each non-straight tableau T or $T' \circ T''$ and then choosing $\underline{a}$ and $\underline{b}$ as in Algorithm Row-Straighten. The above theorem does not require that $\mathscr{L} = \mathscr{L}^-$, although one requires the notion of a non-commutative Groebner basis for general $\mathscr{L}$.

Specializing to the case $\mathscr{L} = \mathscr{L}^-$, it is possible to restrict $r + s$ to be one more than the maximum of the number of columns in T_i and the number of columns in T_j while simultaneouly eliminating any other restrictions on the indices $\underline{a}$ and $\underline{b}$; see [T] for the underlying straightening law together with applications to quantum Schur modules. In general, that straightening law fails when $\mathscr{L}$ contains positive letters.

Theorem 7.6 also holds if in Definition 7.3, we defined $\prod_{i=1}^{k} T_i' < \prod_{j=1}^{k} T_j''$ when either $w_{T_1' \circ \cdots \circ T_k'}$ is smaller in lexicographic order than $w_{T_1'' \circ \cdots \circ T_k''}$, or these words are equal and $c_{T_1' \circ \cdots \circ T_k'}$ is smaller in lexicographic order than $c_{T_1'' \circ \cdots \circ T_k''}$.

One could also reduce the variables used in Theorem 7.6 to the set of all shape D straight tableaux on $\mathscr{L}$. This eliminates the need for the linear relations (7), but the new quadratic relations are significantly more complicated.

Porism 7.7. The monomial $T_1 \cdot T_2 \cdots T_k$ is standard with respect to the Groebner basis in Theorem 7.6 iff $T_1 \circ T_2 \circ \cdots \circ T_k$ is straight.

The existence of a degree 2 Groebner basis for an algebra is known to imply that there is an (infinite) linear free resolution of the ground field over the algebra.

A SAGBI (subalgebra analogue of a Groebner basis for ideals) basis, see [KaMa89, RoSw90], is a generating set for a subalgebra such that the initial terms of the subalgebra are contained in the algebra generated by the initial terms of the generating set.

THEOREM 7.8. *Let D be a row-convex shape. The straight basis elements of shape D form a SAGBI basis for R^D with respect to any diagonal term order.*

Proof. It suffices to show that if p is in R^D then its initial term is the product of the initial terms of some multiset of straight tableaux. Define $D^{\circ k}$ to be the shape formed by replacing each row of D with k copies of itself. So init(p) is init(p_k) where p_k is the component of p lying in $\mathscr{S}^{D^{\circ k}}$ with k maximal such that $p_k \neq 0$. Since the initial terms of straight tableaux of fixed shape are distinct, init$(p) = [T]$ for some straight tableaux T of shape $D^{\circ k}$. But we can write $T = T_1 \circ \cdots \circ T_k$. Each T_i must be straight and init$([T]) = \prod_{i=1}^{k}$ init$([T_i])$. ∎

COROLLARY 7.9. *Let D be a row-convex shape. The row-standard tableaux of shape D form a SAGBI basis for R^D with respect to any diagonal term order.*

By the usual results these SAGBI bases give algorithms for determining whether a polynomial in variables $x_{i,j}$ belongs to $R^D(\mathscr{L})$ or (*a forteriori*) $\mathscr{S}^D(\mathscr{L})$ and, if so, writing it in terms of the generators $[T]$ where T is straight of shape D. By results of [Stu96], a SAGBI basis for an algebra allows that algebra to be deformed to an algebra generated by monomials. In [T97a] this deformation is used to prove that the subalgebra generated by all tableaux of a fixed row-convex shape is Cohen–Macaulay; further applications to questions of Cohen–Macaulayness appear in [T00b].

8. A BRANCHING RULE AND FLAGGED CORNER-CELL RECURRENCE

Our final application concerns a branching rule for row-convex representations. The Schur and Weyl modules $\mathcal{S}^D(\mathcal{L}^-)$ and $\mathcal{S}^D(\mathcal{L}^+)$ are GL_n representations with GL_n-action induced by the algebra homomorphism $g: Super([\mathcal{L} \mid \mathcal{P}]) \to Super([\mathcal{L} \mid \mathcal{P}])$ given by $g((r \mid s)) = \sum_i g_{i,r}(i \mid s)$ where $g \in GL_n$ equals $(g_{i,j})$. In order to handle sets $\mathcal{L}$ containing letters of both positive and negative sign, we will work with representations of the general linear Lie superalgebra, $pl_{\mathcal{L}}$.

We express a $pl_{\mathcal{L}}$-representation, corresponding to a row-convex shape D, in terms of $pl_{\mathcal{L}\setminus\{a\}}$ representations (for some $a \in \mathcal{L}$) corresponding to subshapes of D. The combinatorics for the case $\mathcal{L} = \mathcal{L}^+$ is identical to that of the branching rule in [RS98] and new when $\mathcal{L} = \mathcal{L}^-$. We present a filtration that realizes this branching rule in a characteristic-free fashion. This provides the row-convex case of filtration conjectured in [RS98] to exist for all %-avoiding tableaux. It should be noted that the orientations of [RS98] are at variance from those of [RS95]; we adhere to the orientation of the latter. Thus the term row-convex in [RS98] should be read as "column-convex" in the context of both [RS95] and the present paper. The branching rule presented below generalizes to the case of flagged super-Schur modules; branching rules for flagged Schur modules are not treated in [RS98].

First we construct the general linear Lie superalgebras following Scheunert [Sc79].

A free **Z**-module F is *signed* when it has distinguished free submodules F_0 and F_1 whose direct sum is F. Elements of F_0 and F_1 are called homogeneous and $|x| = i$ for $x \in F_i$.

A free signed **Z**-module is a *Lie superalgebra* when it is endowed with a *superbracket* $[\,,]$ satisfying the commutativity relation,

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

for homogeneous elements x, y and the super-Jacobi identity

$$(-1)^{|a||c|}[a,[b,c]] + (-1)^{|a||b|}[b,[c,a]] + (-1)^{|b||c|}[c,[a,b]] = 0$$

for homogeneous elements a, b, c.

Following [BT91], the general linear Lie superalgebra $pl_{\mathcal{L}}$, associated to the signed alphabet, $\mathcal{L}$, is the vector space (over **Q**) with basis $E_{a,b}$ for a, $b \in \mathcal{L}$, where $|E_{a,b}| = |a| + |b|$ and the bracket is

$$[E_{a,b}, E_{c,d}] = \delta_{b,c}E_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{d,a}E_{c,b}.$$

We next describe an action of $E_{a,b}$ on $Super([\mathcal{L} \mid \mathcal{P}])$.

A (left) *superderivation* D on a superalgebra A is a $\mathbf{Z}$-linear endomorphism of A such that for p, q homogeneous in the $\mathbf{Z}_2$ grading of A, the identity $D(pq) = (Dp)q + (-1)^{\epsilon|p|}p(Dq)$ holds for some fixed $\epsilon \in \mathbf{Z}_2$. This ϵ is the *sign* of D, written $|D|$.

We define the *letter polarization*

$$D_{a,b} : Super_{\mathbf{Q}}([\mathscr{L} | \mathscr{P}]) \to Super_{\mathbf{Q}}([\mathscr{L} | \mathscr{P}])$$

to be the superderivation with sign $|a| + |b|$ such that $D_{a,b}(c \mid p) = \delta_{b,c}(a \mid p)$ where δ is the Kronecker delta. It is easy to check that these superderivations are well-defined on the $\mathbf{Z}$-subalgebra $Super([\mathscr{L} | \mathscr{P}])$.

The next example describes the action of the polarization operators in the case that the biproduct is the determinant of a minor.

EXAMPLE 8.1. If $\mathscr{L} = \mathscr{L}^-$ and $\mathscr{P} = \mathscr{P}^-$, then $Super([\mathscr{L} | \mathscr{P}])$ is isomorphic to $\mathbf{Z}[x_{i,j} : i \in \mathscr{L}, j \in \mathscr{P}]$. The action of $D_{i,j}$ (respectively ${}_{i,j}R$) on this algebra is given by $\sum_{p \in \mathscr{P}} x_{i,p}(\partial / \partial x_{j,p})$ (respectively $\sum_{l \in \mathscr{L}} x_{l,i}(\partial / \partial x_{l,i})$).

To make our results characteristic free, we work over $U(pl_{\mathscr{L}})$, the $\mathbf{Z}$-subalgebra of the universal enveloping superalgebra of $pl_{\mathscr{L}}$ generated by all $E_{a,b}$, by $E^i_{a,b}/i!$ for all $i \in \mathbf{N}$ and all $a \neq b$ such that $|a| = |b|$, and by all $\binom{E_{a,a}}{i}$, for $i \in \mathbf{N}$.

PROPOSITION 8.1. *The map $E_{a,b} \mapsto D_{a,b}$, provides a representation of $U(pl_L)$ on $Super([\mathscr{L} | \mathscr{P}])$.*

If D is a shape such that any letter appearing in $Der^-(D)$ appears in $\mathscr{P}$, then this action descends to an action on $\mathscr{S}^D(\mathscr{L})$.

Proof (*Sketch*). Let "1" be a positive letter. Define the polarization operators to act as superderivations on $Super(\mathscr{L})$. The action is that given by identifying $Super(\mathscr{L})$ with $Super([\mathscr{L} | \{1^+\}])$. It can be shown that the polarization operators satisfy $D_{a,b}(p \mid q) = (D_{a,b}p \mid q)$ where $(\,|\,)$ is the bilinear form of Definition 2.2. It is then clear that $\mathscr{S}^D(\mathscr{L})$ is closed under the action of the superderivations.

Details may be found in [T97a]. ∎

The branching rule for $\mathscr{S}^D$ involves removing vertical or horizontal strips from D.

DEFINITION 8.2. Let D be a sorted row-convex shape. Define a *horizontal strip*, E^+, in D to be any subset of the cells of D such that there exists a shape D straight tableau, T, on some alphabet $a^+ < b_1 < b_2 < \cdots$ where the cells in T that contain a^+ are precisely the cells of E. Similarly, define a *vertical strip*, E^- as any set of cells containing all the negative letters a^- appearing in some straight tableau of shape D on some alphabet $a^- < b_1 < b_2 < \cdots$.

Let $\underline{g}$, $\underline{f}$ be two weakly increasing sequences of letters indexed by the elements of $\mathscr{P}$. A vertical or horizontal strip is a-flagged (with respect to $\underline{g}$, $\underline{f}$) if it contains cells only in columns i where $g_i \leq a \leq f_i$.

Note that strips are allowed to be empty.

LEMMA 8.3. *Let E be a vertical (respectively horizontal) strip in a row-convex shape D. Let $\mathscr{I}_E$ be the multiset of column indices indicating in which columns the cells of the strip appear. If E' is another vertical (respectively horizontal) strip, then $\mathscr{I}_E = \mathscr{I}_{E'}$ implies $E = E'$.*

Proof. We utilize an alternative to Algorithm Straight-Filling for producing straight tableau with specified column content and shape. Suppose D has n cells. Define the desired contents of the columns of a tableau by a biword $\mathbf{u} = \left(\begin{smallmatrix} \cdots & \hat{\underline{u}} & \cdots \\ \cdots & \check{\underline{u}} & \cdots \end{smallmatrix}\right)$ where $\underline{\hat{u}} = w_{Der^-(D)}$ and $\check{u}_i <^- \check{u}_{i+1}$ if $\hat{u}_i = \hat{u}_{i+1}$. Define a biword $\left(\begin{smallmatrix} \cdots & \hat{\underline{w}} & \cdots \\ \cdots & \check{\underline{w}} & \cdots \end{smallmatrix}\right) = \left(\begin{smallmatrix} \hat{u}_{\sigma(1)}, \dots, \hat{u}_{\sigma(n)} \\ \check{u}_{\sigma(1)}, \dots, \check{u}_{\sigma(n)} \end{smallmatrix}\right)$ by permuting the entries of $\mathbf{u}$ so that $\underline{\check{w}}$ weakly increases and so $\check{w}_i = \check{w}_{i+1}$ implies $\hat{w}_i \leq \hat{w}_{i+1}$; when $|\check{w}_i| = 0$, this inequality is strict.

Using this biword $\mathbf{w}$, we fill the tableau by starting with an empty tableau of shape D and adding successive letters reading left to right through the biword. At step j we place $\check{w}_j$ in the northmost available cell (say row i) in column $\hat{w}_j$ such that either $(i, \hat{w}_j - 1)$ is not in the diagram or such that the cell $(i, \hat{w}_j - 1)$ contains a letter x with $x <^+ \check{w}_j$. If no such cell exists, then the biword does not arise from a straight tableau of the given shape. To verify this algorithm, observe that if we do not put the value $\check{w}_j$ into $(i, \hat{w}_j)$, then $\check{w}_j$ appears in $(i', \hat{w}_j)$ for some $i' > i$, so either row-standardness is violated or we have created a flippable inversion in cells $(i, \hat{w}_j)$ and $(i', \hat{w}_j)$.

The lemma is an immediate consequence of the algorithm's correctness. ∎

DEFINITION 8.4. Suppose that D is a row-convex shape, $\mathscr{L}$ is an alphabet, and $\mathbf{Z}[t_l : l \in \mathscr{L}]$ is a polynomial ring. Let $\underline{g}$, $\underline{f}$ be two weakly increasing sequences in $\mathscr{L}$ indexed by the elements of $\mathscr{P}$. Define

$$ch^D_{\underline{g}, \underline{f}}(\mathscr{L}) = \sum_T \prod_{(i,j) \in D} t_{T_{(i,j)}},$$

where the sum runs over all $\underline{g}$, $\underline{f}$-doubly flagged straight tableaux T of shape D on $\mathscr{L}$ and where $T_{(i,j)}$ is the (i, j)th entry of T.

When $\underline{f}$ and $\underline{g}$ are trivial, that is, they contain respectively only the largest and smallest elements of $\mathscr{L}$, and when $\mathscr{L}$ contains letters of only one sign, this is the formal character of the $GL(|\mathscr{L}|)$-representation $\mathscr{S}^D(\mathscr{L})$. If just one of $\underline{f}$, $\underline{g}$ is trivial, we get the formal character of a representation of a Borel subgroup.

The following identity is immediate from the definition of a straight tableau.

PROPOSITION 8.5. *Fix two weakly increasing sequences $\underline{g}$, $\underline{f}$ of letters, and let a be the minimal element of $\mathscr{L}$. If D is a sorted row-convex diagram, then*

$$ch^{D}_{\underline{g},\underline{f}}(\mathscr{L}) = \sum_{E} ch^{D/E}_{\underline{g},\underline{f}}(\mathscr{L} \setminus \{a\}),$$

where the sum runs over all a-flagged horizontal (respectively vertical) strips E in D when $a \in \mathscr{L}$ is positive (respectively negative), and where D/E is the diagram formed by removing E from D.

The case where $\underline{f}$, $\underline{g}$ are trivial and $\mathscr{L} = \mathscr{L}^{+}$ above is due to [RS98].

Preparatory to establishing a filtration for $pl_{\mathscr{L}}$-modules $\mathscr{S}^{D}(\mathscr{L})$ that realizes this identity we define some components of that filtration.

DEFINITION 8.6. If E, E' are two vertical (or two horizontal) strips in D, define $E < E'$ in dominance order when for all i, the number (counted with multiplicity) of elements in $\{1, \ldots, i\}$ in $\mathscr{I}_E$ is at least as large as the number of times these elements appear in $\mathscr{I}_{E'}$. Similarly, we write $\mathscr{I}_E < \mathscr{I}_{E'}$.

Let E be a vertical (respectively horizontal) strip and let $a \in \mathscr{L}$ be negatively (respectively positively) signed. Define

$$\mathscr{S}^{D, \geq E}(\mathscr{L}; a) = \operatorname*{span}_{T} {}_{\mathbf{Z}}\{[T]\},$$

where T runs over all shape D tableaux on $\mathscr{L}$ in which a appears in a vertical strip E' weakly dominating E.

Define $\mathscr{S}^{D, > E}(\mathscr{L}; a)$ identically except for the requirement that E' must strictly dominate E.

It is immediate from the definition that $\mathscr{S}^{D, \geq E}(\mathscr{L}; a)$ and $\mathscr{S}^{D, > E}(\mathscr{L}; a)$ are $pl_{\mathscr{L} \setminus \{a\}}$-representations.

THEOREM 8.7. *Let D be a row-convex shape. If a is a negatively (respectively positively) signed letter in $\mathscr{L}$ and E is a horizontal (respectively vertical) strip in D, then*

$$\mathscr{S}^{D, \geq E}(\mathscr{L}; a) / \mathscr{S}^{D, > E}(\mathscr{L}; a) \simeq \mathscr{S}^{D/E}(\mathscr{L} \setminus \{a\})$$

as a $pl_{\mathscr{L} \setminus \{a\}}$-representation. Here D/E is the shape formed by removing E from D.

Proof. To check that the left-hand side gives a representation with the same formal character as the right-hand side, it suffices to observe that given a row-standard tableau T such that E comprises the cells occupied by a, then for any tableau appearing in the straightened form of $[T]$, the cells occupied by a form a strip E' determined by a multiset $I' \geq I$. This can be seen by directly examining the straightening relations. In particular, any straightening relation which moves the a's produces a row-standard tableau in which the a's form a horizontal (respectively vertical) strip indexed by some $I' > I$.

For any tableau T in which the a's appear in the strip E, the isomorphism sends $[T]$ to $(-1)^{n_T}[T/E]$ where T/E is the tableau T with the cells of E (equivalently the a's) deleted and where

$$n_T = (|a| + 1) \sum_{\substack{(r,c) \in D \setminus E \\ T_{r,c} \in \mathscr{L}^+}} \#a\text{'s appearing in rows} \leq r$$

We verify that this map is indeed an isomorphism of $gl_{\mathscr{L} \setminus \{a\}}$-modules as follows. Consider the superalgebra $Super([\mathscr{L} \mid \mathscr{P}])$ and mod out by the $pl_{\mathscr{L} \setminus \{a\}}$-submodule $\mathscr{M}_E$ of all monomials not divisible by $\prod_{i \in \mathscr{I}_{\mathscr{E}}} (a \mid i)$. Call the quotient map π_1. Let $\mathscr{S}'$ be $\pi_1(\mathscr{S}^{D, \geq E}(\mathscr{L}; a))$. By considering the initial terms of the straight basis for $\mathscr{S}^{D, \geq E}(\mathscr{L}; a)$, it is easy to check that the kernel of the above projection onto $\mathscr{S}'$ is $\mathscr{S}^{D, > E}(\mathscr{L}; a)$. So it suffices to show that $\mathscr{S}'$ is isomorphic to the right-hand side of the isomorphism in the statement of the theorem.

It is immediate that the quotient of the $Super([\mathscr{L} \mid \mathscr{P}])$ by $\mathscr{M}_E$ is isomorphic as a $pl_{\mathscr{L} \setminus \{a\}}$-module to $Super([\mathscr{L} \setminus \{a\} \mid \mathscr{P}])$ under the map $\pi_2 : M \cdot \prod_{i \in \mathscr{I}_{\mathscr{E}}} (a \mid i) + \mathscr{M}_E \mapsto M$ where M is any monomial in $Super([\mathscr{L} \setminus \{a\} \mid \mathscr{P}])$. It remains to check that the image of this map applied to $\mathscr{S}'$ is $\mathscr{S}^{D/E}(\mathscr{L} \setminus \{a\})$.

The isomorphism described above is $\pm \pi_2 \circ \pi_1$. The sum in the formula for n_T appears from commuting negatively signed biproducts as the $(a \mid i)$'s are commuted to the right end of the monomial they lie in. The overall factor of ± 1 appears when a is positive and is the sign of the permutation required to permute the $(a \mid i)$'s from the strictly increasing order of $\prod_{i \in \mathscr{I}_{\mathscr{E}}} (a \mid i)$ to the order in which they appear in some (any) monomial of $\pi_1([T])$ for some (any) tableau T whose a's appear in the horizontal strip E. ∎

The interested reader can also verify that $[T] \mapsto (-1)^{n_T}[T/E]$ is a $pl_{\mathscr{L} \setminus \{a\}}$-map directly. Showing that the action of the map on any particular $[T]$ commutes with $pl_{\mathscr{L} \setminus \{a\}}$-action is immediate from the sign rules. One then analyzes the identities in Proposition 5.3 and shows that those used to produce the straightening law are preserved under the map $[T] \mapsto (-1)^{n_T}[T/E]$ and then employs equivariance under $pl_{\mathscr{L} \setminus \{a\}}$-action to finish showing that $[T] \mapsto (-1)^{n_T}[T/E]$ is linear.

The first half of the proof of Theorem 8.7 amounts to proving a restriction on the allowable contents of a tableau appearing in the straightening of $[T]$. A more sophisticated result along these lines is proved in [T97a, Chap. III, Sect. 6].

COROLLARY 8.8. *Let D be a row-convex shape and let a be a negatively (respectively positively) signed letter in $\mathscr{L}$. Let $E_1, \ldots, E_k$ be all vertical (respectively horizontal) strips in D ordered compatibly with dominance, so that $i > j$ implies $E_j \ngeq E_k$. The filtration*

$$\begin{aligned}\mathscr{S}^D &= \sum_{i=1}^{k} \mathscr{S}^{D, \geq E_i}(\mathscr{L}; a) \supseteq \cdots \supseteq \sum_{i=j}^{k} \mathscr{S}^{D, \geq E_i}(\mathscr{L}; a)\\ &\supseteq \cdots \supseteq \mathscr{S}^{D, \geq E_k}(\mathscr{L}; a) \supseteq 0\end{aligned}$$

has $\mathscr{S}^{D/E_j}(\mathscr{L} \setminus \{a\})$ as the quotient, up to $pl_{\mathscr{L}\setminus\{a\}}$-isomorphism, of its jth term by its $j+1$st term.

If $\mathscr{L}$ is sufficiently large, then the containments in the above filtration are all strict.

The preceding results generalize immediately to the S_n-representations provided by the Specht modules.

If we define $\mathscr{S}_f^{D, \geq E_i}(\mathscr{L}; a)$ to be the flagged super-Schur module found by taking the image of $\mathscr{S}_f^{D, \geq E_i}(\mathscr{L}; a)$ under the map $(l \mid p) \mapsto 0$ when $l > f_p$, then we have the following.

PROPOSITION 8.9. *Maintaining the notation of Corollary 8.8, the filtration*

$$\begin{aligned}\mathscr{S}_f^D &= \sum_{i=1}^{k} \mathscr{S}_f^{D, \geq E_i}(\mathscr{L}; a) \supseteq \cdots \supseteq \sum_{i=j}^{k} \mathscr{S}_f^{D, \geq E_i}(\mathscr{L}; a)\\ &\supseteq \cdots \supseteq \mathscr{S}_f^{D, \geq E_k}(\mathscr{L}; a) \supseteq 0\end{aligned}$$

of $\mathscr{S}_f^D$ by B-modules has $\mathscr{S}_f^{D/E_j}(\mathscr{L} \setminus \{a\})$ as the quotient, up to isomorphism, of its jth term by its $j+1$st term. Here B is the subalgebra of $U(pl_L)$ generated by all $E_{b,a}^i/i!$ for $b > a$ and all $\binom{E_{a,a}}{i}$.

When $\mathscr{L} = \mathscr{L}^-$, these results generalize to quantum Schur modules; details appear in [T].

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